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The Role of Sanctioning in the Evolutionary Dynamics of Collective Action

Vítor Vasco Lourenço de Vasconcelos

Dissertação
Mestrado em Física
(Especialização em Física Estatística e Não Linear)

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The whole is more than the sum of its parts.
Aristoteles

Abstract

Preventing global warming requires overall cooperation. Contributions will depend on the risk of future losses, which plays a key role in decision-making. Here, I discuss an evolutionary game theoretical model in which decisions within small groups under high risk and stringent requirements toward success significantly raise the chances of coordinating to save the planet's climate, thus escaping the tragedy of the commons. I analyze both deterministic dynamics in infinite populations, and stochastic dynamics in finite populations.

I also study the impact of different types of sanctioning mechanisms in deterring non-cooperative behavior in climate negotiations, towards the mitigation of the effects of climate change. To this end, I introduce punishment in the collective-risk dilemma and study the dynamics of collective action in finite populations. I show that a significant increase in cooperation is attained whenever individuals have the opportunity to contribute (or not) to institutions that punish free riders. I investigate the impact of having local instead of global sanctioning institutions, showing that the former – which are expected to require less financial resources and which involve agreements between a smaller number of individuals – are more conducive to the prevalence of an overall cooperative behavior.

In the optics of evolutionary game theory, the system dynamics is solved by means of a multidimensional stochastic Markov process. The interaction between individuals is not pairwise but it occurs in n-person games; the individuals have perception of the risk and are allowed to explore unpopulated strategies.

Keywords: Emergence of Cooperation; Evolution of Cooperation and Institutions; Collective Action; Tragedy of the Commons; Climate Change; Sustainability Science; Complex Systems; Stochastic Systems; Evolutionary Game Theory; Evolutionary Dynamics.

Resumo

A prevenção do aquecimento global requer cooperação global. As atuais contribuições dependerão do risco das perdas futuras, desempenhando assim um papel fundamental na tomada de decisões. Nesta tese discuto um modelo teórico para um jogo evolutivo, no qual a tomada de decisões envolvendo grupos pequenos, com alto risco e requisitos rigorosos em direção ao sucesso aumenta significativamente as hipóteses de coordenação para a salvação do clima do planeta, evitando assim a tragédia dos comuns. Tanto a dinâmica determinística em populações infinitas como a dinâmica estocástica em populações finitas são analisadas.

Além disto, estudo ainda o impacto de diferentes tipos de mecanismos de sanção para desencorajar o comportamento não cooperativo nas negociações climáticas, de forma a mitigar os efeitos das alterações climáticas. Para este fim, introduzo punição no jogo evolutivo e estudo a dinâmica da ação coletiva em populações finitas. Mostro que um aumento significativo na cooperação é alcançado quando os indivíduos têm a oportunidade de contribuir (ou não) para as instituições que punem os chamados "free riders". Investigo o impacto da conceção de instituições fiscalizadoras locais em vez de instituições globais, mostrando que as primeiras – das quais se espera que exijam menos recursos financeiros e que envolvam acordos entre um número menor de indivíduos – são mais favoráveis para a prevalência de um comportamento global cooperativo.

Na ótica da teoria dos jogos evolutiva, a dinâmica do sistema é resolvida por meio de um processo estocástico de Markov multidimensional. A interação entre indivíduos não é entre pares, mas ocorre em jogos de n pessoas, os indivíduos têm perceção do risco e têm a capacidade de explorar estratégias despovoadas.

Palavras Chave: Emergência da Cooperação; Evolução da Cooperação e Instituições; Ação Coletiva; Tragédia dos Comuns; Alterações Climáticas; Ciências da Sustentabilidade; Redes Complexas; Sistemas Complexos; Teoria de Jogos Evolutiva; Dinâmica Evolutiva.

Contributions

Bellow follows a list of the manuscripts related with the present work carried out in this thesis.

- **V.V. Vasconcelos**, J.M. Pacheco and F.C. Santos,
  Local sanctioning institutions efficiently promote cooperation in the governance of risky commons
  (2012)(Submitted)

- F.C. Santos, **V.V. Vasconcelos**, M.D. Santos, P.B. Neves, J.M. Pacheco
  Evolutionary dynamics of climate change under collective-risk dilemmas
  Mathematical Models and Methods in Applied Sciences (M3AS), **22** Suppl., 1140004 (2012)

- **V.V. Vasconcelos**, F. Raischel, M. Haase, J. Peinke, M. Wächter, P.G. Lind, D. Kleinhans
  Principal axes for stochastic dynamics
  Physical Review E, 84, 031103 (2011)

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Chapter 1

Introduction

“There is little time left. The opportunity and responsibility to avoid catastrophic climate change is in your hands”, said the United Nations Secretary-General to nearly 100 world leaders, on a Climate Change summit by the end of 2009, before the Copenhagen summit. By now, time is still running out.

In a dance that repeats itself cyclically, countries and citizens raise significant expectations every time a new International Environmental Summit is settled. Unfortunately, few solutions have come out of these colossal and flashy meetings. This represents a challenge on our current understanding of models on decision-making: more effective levels of discussion, agreements and coordination must become accessible. From Montreal and Kyoto to Copenhagen summits, it is by now clear how difficult it is to coordinate efforts [1, 2]. Copenhagen agreement was, in first place, intended to extend Kyoto's caps to the US, China, India and other expanding economies. However, this approach was clearly heading towards a dead end [3].

Climate is a public good, and, probably, the welfare of our planet accounts for the most important and paradigmatic example of a public good: a global good from which every single person profits, whether she contributes or not to maintain it. However, these summits failed to recognize the well-studied difficulties of cooperation in public-good games [4, 5, 6]. Often, individuals, regions or nations opt to be free riders, hoping to benefit from the efforts of others while choosing not to make any effort themselves. Most cooperation problems faced by humans share this setting, in which the immediate advantage of free riding drives the population into the tragedy of the commons [4], the ultimate limit of widespread defection [4, 5, 6, 7, 8, 9, 10, 11]. When dealing with such an essential public good as climate, many efforts are made to avoid this. To avoid free riding is then a major aim to the countries, so that efforts are shared for all and balanced measures can then be taken. The strive to identify and improve the mechanisms that allow this will be the goal of this work.

One of the multiple fatal flaws often appointed to such agreements is a deficit in the overall perception of risk of widespread future losses, in particular the perception by those occupying key positions in the overall political network that underlies the decision process [10, 12, 13]. Another problem relates to the lack of sanctioning mechanisms to be imposed on those who do not contribute (or stop contributing) to the welfare of the planet [2, 14, 15]. Moreover, agreeing on the way punishment should be implemented is far from reaching a consensus, given the difficulty in converging on the *pros* and *cons* of some procedures against others. Many possibilities have been under consideration - from financial penalties, trade sanctions, to emissions penalties under future climate change agreements – but their details have not been well established and negotiations are usually slow and difficult [16]. The impasse over these measures is expected since their consequences do not have a solid theoretical or even experimental background.

To address this and other cooperation conundrums, ubiquitous at all scales and levels of complexity, the last decades have witnessed the discovery of several core mechanisms responsible to promote and maintain cooperation at different levels of organization [4, 6, 11, 17, 18, 19, 20, 21, 22, 23, 24, 25, 26, 27, 28].

Most of these key principles have been studied within the framework of two-person dilemmas, such as the Prisoner's Dilemma, which constitute a powerful metaphor to describe conflicting situations often encountered in the natural and social sciences. Many real-life situations, however, are associated with collective action based on joint decisions made by a group involving more than two individuals [4, 6, 17, 29]. These types of problems are best dealt-with in the framework of n-person dilemmas and Public Goods games, involving a much larger complexity that only recently started to be unveiled [6, 30, 24, 31, 32, 33, 34, 35].

In this thesis, I aim to model the decision making process as a dynamical process, in which behaviours evolve in time [24, 36], taking into consideration decisions and achievements of others, which influence one's own decisions [37, 38, 39]. I implement such behavioural dynamics in the framework of Evolutionary Game Theory, in which the most successful (or fit) behaviours will tend to spread in the population. This way, one is able to describe strategic interactions between individuals, complemented by evolutionary principles. In particular, I will do so in finite populations, where such fitness driven dynamics occurs in the presence of errors (leading to stochastic effects), both in terms of errors of imitation [40] as well as in terms of behavioural mutations [41], the latter accounting for spontaneous exploration of the possible strategies.

In Chapter 2 I will formally define the system of variables and introduce some useful concepts for the analysis. The following chapter describes the standard approach of evolutionary game theory. Chapter 4 shows the effect of group size and risk awareness in the decision making process in the context of the Collective Risk Dilemma (CRD), a simple *Public Goods game* that mimetizes the problem at stake [7, 2, 10, 12]. Then, Chapter. 5 presents the effects of punishment via institutions when playing against defectors (which leads to higher-order cooperation dilemmas [42, 22, 43, 24]). Finally, in Chapter 6, I will bring the three strategies together and show that, in the presence of risk, sanctioning needs to be neither high nor does it require a global institution to supervise abidance (or opposition) to the agreement. Instead, multiple local institutions [14] may very well provide an easier solution to this "game that concerns all of us, and we cannot afford to lose". Chapter 7 closes the thesis with further discussion and conclusions.

Chapter 2

Framework

Most of the phenomena in nature evolve in a very complicate and irregular way. For sure, population and decision making processes are so intricate that models have no hope in computing all variations and variables in detail. Even if one would try, and succeed, the results would be so cumbersome that, by themselves, would be imperceptible. What is of use is how average properties evolve and are established and, often, this is described by much simpler laws. For this reason, I will make use of a variety of concepts related to the theory of probability and statistics.

In this chapter the framework used throughout the text is presented. Most of the notation is defined also here. From stochastic processes to the M-equation, this chapter sets itself as an overview and collection of results that are relevant further on.

2.1 Stochastic Processes

Individuals in large populations are often named, or even categorized, according to their preferences, behaviors, physical and psychological resemblance and many other characteristics that make people alike. This can happen for the most various reasons, from hierarchisation to discrimination, but is often a way to simplify the line of thought by means of despising individuality. In what climate is concerned, people, nations, leaders and so on are classified according to their acts, to their strategy on climate issues.

Therefore, it is convenient to consider a set of N elements, each of these can be in one of $s+1$ states. I will think on the elements as the individuals of a population who can adopt different strategies: $S_1, ..., S_{s+1}$. Let $i_k(t)$ be, at a given time t, the number of individuals with the strategy S_k, where $k = 1, ..., s+1$. The number of individuals of a given type will randomly evolve in time according to some rule. To study these quantities I need the proper framework; the probability theory has the right objects.

Let i be a *random* or *stochastic variable* defined by a set Ω_i and a function $P_i(x)$. Ω_i represents the set of possible states of i; I will be calling it indiscriminately "range", "sample space", "phase space" or "domain". $P_i(x)$ is a probability distribution defined over Ω_i. However, when studying a system, one needs more than just a single value of its properties. One is usually interested in describing their evolution in time. In order to do so, one needs more than one stochastic variable. Suppose I build a set according to Def.(1). It depends on the random variable i and on a parameter, t, that represents time.

Definition 1 Stochastic Process

Let Y be a a set of stochastic variables indexed by a parameter, t: i_{t_1}, i_{t_2}, ..., i_{t_m}. If t represents time, Y is a Stochastic Process. Whenever t does not represent time, Y is called a *random function*.

This set can be written in a compact way which easily takes into account all possible values of the indexing parameter: $Y_i(t)$. To cut out notation, often the set is named after the stochastic variable, $i_i(t)$ or, simply, $i(t)$. With the given care, one can write

$$\begin{aligned} Y_i(t) &:= \{f(i,t_1),...,f(i,t_m)\} \\ &\equiv \{i_{t_1};...;i_{t_m}\} \\ &\equiv i_i(t) \\ &\equiv i(t). \end{aligned} \tag{2.1}$$

All i_k, mentioned before as the number of individuals with a given strategy, perfectly fit in this definition and so can be treated as a stochastic process. Furthermore, note that

$$i_1 + ... + i_{s+1} = N. \tag{2.2}$$

Since Eq.(2.2) determines, for example, i_{s+1} in terms of the remaining i_k, I only need to retain s of them. Strictly speaking, one defines $\mathbf{i}(t) = \{i_1,...,i_s\}$ as multivariable stochastic process over an s-dimensional sample space, $\Omega_{\mathbf{i}}$, given in Eq.(2.3).

$$\begin{aligned} &i_1 \in \{0,1,...,N\} \\ &i_2 \in \{0,1,...,N-i_1\} \\ &... \\ &i_s \in \{0,1,...,N-i_1-...-i_{s-1}\} \end{aligned} \tag{2.3}$$

Within the framework of stochastic processes, there is a specific subclass of processes called Markov processes. Markov processes are, by far, the most important, well known and used stochastic processes. In part due to their manageability but, ultimately, because any isolated system can be described as a Markov process once one considers all microscopic variables. Evidently, this is not always possible or even desirable. The task is to find a *small* collection of variables with the Markov property at a given time scale.

These are formally defined in Def.(2), which states that the conditional probability of a future event at t_n, $P_{1|n}$, is independent of the knowledge of the values at times earlier than t_{n-1}. Informally, one can say that the information required to compute future statistical properties depends only on the actual state; the system has no memory.

Definition 2 Markov Property

Let $\mathbf{i(t)}$ be a Markov process, indexed by a set of n successive times, i.e. $t_1 < t_2 < ... < t_n$. The conditional probability of getting $\mathbf{i}_n$ at t_n, given the set of observed values $\mathbf{i}_{n-1}$ at t_{n-1}, ..., $\mathbf{i}_1$ at t_1 is given by:

$$P_{1|n-1}(\mathbf{i}_n, t_n|\mathbf{i}_1, t_1; ...; \mathbf{i}_{n-1}, t_{n-1}) = P_{1|1}(\mathbf{i}_n, t_n|\mathbf{i}_{n-1}, t_{n-1}).$$

The evolution of a population is not a purely random process: there must be mechanisms that generate a global tendency for individuals. In a practical point of view, Markov property is the simplest way of introducing statistical dependence into the models built and, hence, such tendency.

Individuals are part of a population with a given configuration of strategies and they can opt to change their own whenever they feel their outcome is not the best: they compare themselves with the present situation and choose a better strategy. Therefore, one supposes that, at a give state, the evolution of the system depends only on the present configuration so that $\mathbf{i}(t) = \{i_1, ..., i_s\}$ is a Markov process.

The study of a Markov process consists in determining its probability density function (PDF) evolution, $p_{\mathbf{i}}(t)$. Since $\mathbf{i}(t)$ has the Markov property, its transition probability $P_{1|1}$ respects the discrete time M-Equation, Eq.(2.4), and consequently, with a delta-shaped initial condition, its PDF also respects it [44]. This is a gain-loss equation that allows one to compute $p_{\mathbf{i}}(t)$ given the transition probability from the configuration $\mathbf{i}$ to the configuration $\mathbf{i}'$, $T_{\mathbf{i}'\mathbf{i}}$.

$$p_{\mathbf{i}}(t+\tau) - p_{\mathbf{i}}(t) = \sum_{\mathbf{i}'} \{T_{\mathbf{i}\mathbf{i}'} p_{\mathbf{i}'}(t) - T_{\mathbf{i}'\mathbf{i}} p_{\mathbf{i}}(t)\} \tag{2.4}$$

As a result, the problem is reduced to the computation of $T_{\mathbf{i}\mathbf{i}'}$. Furthermore, making the left side zero, in the search for a stationary $p_{\mathbf{i}}(t)$, one falls into an eigenvector search problem [44]. This way, my first goal will be to describe a model that can be used to calculate $T_{\mathbf{i}'\mathbf{i}}$ in terms of quantities that can relate to experiments.

2.2 Kramers-Moyal Expansion to Langevin Equation

Before I proceed with calculations on $T_{\mathbf{i}'\mathbf{i}}$, I want to get further insight on its meaning and derive some related quantities. Therefore, in this section I will review some important equations together with their interpretation, which will be of use latter on. By the end of the section, I will derive these equations, in the framework presented so far, in order to relate their well known quantities to $T_{\mathbf{i}'\mathbf{i}}$.

The M-equation, Eq.(2.4), can be rewritten in an equivalent formulation, the Kramers-Moyal (KM) expansion, Eq.(2.5). The functions $D^{(n)}_{i_1,\dots,i_n}(\mathbf{x},t)$ characterize the process and each one is called the n-th Kramers-Moyal coefficient.

$$\frac{\partial p}{\partial t}(t) = \sum_{n=1}^{+\infty}(-1)^n \sum_{i_1,\dots,i_n}^{s} \left[\prod_{l=1}^{n} \frac{\partial}{\partial x_{i_l}}\right] D^{(n)}_{i_1,\dots,i_n}(\mathbf{x})p(\mathbf{x},t) \tag{2.5}$$

This is an equation for the time evolution of the PDF, $p(t)$, of an s-dimensional continuous Markov process, $\mathbf{X}(t)$. It often allows one to make use of perturbation theory and, consequently, the effect of its successive terms can be more easily studied. Hence, the equation containing only the first two terms is well discussed in the literature and is called the Fokker-Planck Equation (FPE), Eq.(2.6). Pawula's theorem reenforces the study of this equation stating that stochastic processes obey this equation not only as a first order approximation but exactly as long as one of the odd KM coefficient is zero [45].

$$\frac{\partial p}{\partial t}(t) = -\sum_{i=1}^{s} \frac{\partial}{\partial x_i}\left[D^{(1)}_i(\mathbf{x},t)p\right] + \sum_{i=1}^{s}\sum_{j=1}^{s} \frac{\partial^2}{\partial x_i \partial x_j}\left[D^{(2)}_{ij}(\mathbf{x},t)p\right] \tag{2.6}$$

The first KM coefficient, $D^{(1)}_i(\mathbf{x},t)$, is called the Drift and the second, $D^{(2)}_{ij}(\mathbf{x},t)$, the Diffusion.

Associated with the FPE (2.6) is a system of s coupled Itô-Langevin equations, which can be written as [45, 46]

$$\frac{d\mathbf{X}}{dt} = \mathbf{h}(\mathbf{X}) + \mathbf{G}(\mathbf{X})\mathbf{\Gamma}(t) \tag{2.7}$$

Here, $\mathbf{\Gamma}(t)$ is a set of s normally distributed random variables fulfilling

$$\langle \Gamma_i(t) \rangle = 0, \quad \langle \Gamma_i(t)\Gamma_j(t') \rangle = 2\delta_{ij}\delta(t-t') \quad . \tag{2.8}$$

These equations drive the stochastic evolution of $\mathbf{X}(\mathbf{t})$. The vectors $\mathbf{h}$ and the matrices $\mathbf{G} = \{g_{ij}\}$ for all $i, j = 1, \ldots, s$ are connected to the local Drift and Diffusion function through

$$D_i^{(1)}(\mathbf{X}) = h_i(\mathbf{X}) \quad \text{and} \tag{2.9}$$

$$D_{ij}^{(2)}(\mathbf{X}) = \sum_{k=1}^{s} g_{ik}(\mathbf{X}) g_{jk}(\mathbf{X}) \quad . \tag{2.10}$$

While the FPE describes the evolution of the joint distribution of the s variables statistically, the system of Langevin equations in (2.7) models individual stochastic trajectories of a system. In Eq. (2.7) the term $\mathbf{h}(\mathbf{X})$, related to the Drift, contains the deterministic part of the macroscopic dynamics, while the functions $\mathbf{G}(\mathbf{X})$, related to Diffusion, account for the amplitudes of the stochastic forces mirroring the different sources of fluctuations due to all sorts of microscopic interactions within the system.

Notice, however, that the $s \times s$ matrix $\mathbf{G}$ cannot be uniquely determined from the symmetric diffusion matrix $\mathbf{D^{(2)}}$ for $s \geq 2$: the number of unknown elements in $\mathbf{G}$ exceeds the number of known elements in $\mathbf{D^{(2)}}$ leading to $s^2 - \frac{1}{2}s(s+1) = \frac{1}{2}s(s-1)$ free parameters. However, a simple method can be used to obtain $\mathbf{G}$ from $\mathbf{D^{(2)}}$ in such a way that a Langevin equation can be extracted from a FPE, but is not unique [47]. Symbolically I will write this particular $\mathbf{G}$ as $\sqrt{\mathbf{D^{(2)}}}$.

Furthermore, in general, the eigenvalues of these matrices indicate the amplitude of the stochastic force and the corresponding eigenvector indicates the direction toward which such force acts, in the Langevin point of view. Even more interesting features, however, can be extracted from the eigenvalues and eigenvectors.

To each eigenvector of the diffusion matrix one can associate an independent source of stochastic forcing Γ_i. In this scope, the eigenvectors can be regarded as defining principal axes for stochastic dynamics. For instance, the vector field aligned at each point to the eigenvector associated to the smallest eigenvalue of matrix $\mathbf{G}$ defines the paths in phase space towards which the fluctuations are minimal. Furthermore, if the corresponding eigenvalues are very small compared to all the other ones at the respective points, the corresponding stochastic forces can be neglected and the system has only $s-1$ independent stochastic forces. In this situation the problem can be reduced in one stochastic variable by an appropriate transformation of variables, since the eigenvectors in one coordinate system are the same as in another one [47].

Now that I allowed some intuition on these equations, I will perform a Kramers-Moyal expansion in order to identify the KM coefficients of our system, namely the Drift and Diffusion. I will be able to prove that the coefficients are sequentially smaller and therefore justify the analysis using FPE and Langevin interpretation.

Consider that all transition probabilities and statistical properties defined so far as function of the number of individuals of the different strategies, $\mathbf{i}$, are redefined as functions of the fraction of individuals in the total population, $\mathbf{X} = \mathbf{i}/N$. Thus, I introduce the changes and notation in Eqs.(2.11). This way, the discrete stochastic process $\mathbf{X}$ will tend to a continuous process as one makes $N \to \infty$ and, therefore, can be described by the FPE and Langevin equation.

Hence, configurations are described by $\mathbf{x}$, with a PDF $\rho(\mathbf{x}, t)$, and $T^{\boldsymbol{\delta}}(\mathbf{x})$ represents a transition between configurations: from the configuration $\mathbf{x}$ in the direction $\boldsymbol{\delta}$ such that it gets to configuration $\mathbf{x}' = \mathbf{x} + \boldsymbol{\delta}$.

$$x_k \equiv \frac{i_k}{N} \tag{2.11a}$$

$$p_{\mathbf{i}}(t) \to p(\mathbf{x}, t) \tag{2.11b}$$

$$T_{\mathbf{i}'=\{i_1+\Delta_1,\ldots,i_s+\Delta_s\}\mathbf{i}=\{i_1,\ldots,i_s\}} \to T^{\boldsymbol{\delta}}(\mathbf{x}) \tag{2.11c}$$

$$\rho(\mathbf{x}, t) \equiv N p(\mathbf{x}, t) \tag{2.11d}$$

Finally, I can rewrite Eq.(2.4) in terms of the PDF of $\mathbf{X}$, $\rho(\mathbf{x}, t)$, as follows.

$$\begin{aligned} \rho(\mathbf{x}, t+\tau) - \rho(\mathbf{x}, t) = \sum_{\boldsymbol{\delta}\neq 0} \Big[&\rho(\mathbf{x}+\boldsymbol{\delta}, t) T^{-\boldsymbol{\delta}}(\mathbf{x}+\boldsymbol{\delta}) - \\ &- \rho(\mathbf{x}, t) T^{\boldsymbol{\delta}}(\mathbf{x}) \Big] \end{aligned} \tag{2.12}$$

Having in mind the aim is a population model, it is reasonable to assume that, as the population increases, the frequency in which an individual of some type changes strategy also increases with N. Thus the update time decreases with $1/N$. Also, at this point, I need not to neglect different individuals simultaneous transitions as long as one guarantees that $\boldsymbol{\delta}$ scales with $1/N$. Then, taking into account that one has two small parameters, τ

and $\boldsymbol{\delta}$, which scale with the same $1/N$ factor and using Eq.(2.13), I expand Eq.(2.12) to the desired order.

$$F(\mathbf{x}+\boldsymbol{\delta}) = \left[\sum_{n=0}^{+\infty} \frac{1}{n!} \sum_{i_1,\ldots,i_n}^{s} \prod_{m=1}^{n} \delta_{i_m} \frac{\partial}{\partial x_{i_m}}\right] F(\mathbf{x}) \tag{2.13}$$

$$\begin{aligned} &= F(\mathbf{x}) + \sum_{k}^{s+1} \delta_k \frac{\partial F}{\partial x_k}(\mathbf{x}) + \\ &+ \frac{1}{2} \sum_{k,l}^{s+1} \delta_k \delta_l \frac{\partial^2 F}{\partial x_k \partial x_l}(\mathbf{x}) + \mathcal{O}(\|\boldsymbol{\delta}\|^3) \end{aligned} \tag{2.14}$$

I start by applying Eq.(2.14) considering $F(t+\tau) = \rho(\mathbf{x}, t+\tau)$ and $F(\mathbf{x}+\boldsymbol{\delta}) = \rho(\mathbf{x}+\boldsymbol{\delta}, t) T^{-\boldsymbol{\delta}}(\mathbf{x}+\boldsymbol{\delta})$. The left side of Eq.(2.12) is easily computed to be

$$\rho(\mathbf{x}, t+\tau) - \rho(\mathbf{x}, t) = \tau \frac{\partial \rho}{\partial t} + \frac{\tau^2}{2} \frac{\partial^2 \rho}{\partial t^2} + \mathcal{O}(\tau^3). \tag{2.15}$$

The right side requires further analysis. Since I already have, to first order in time, a time derivative, I will deduce the complete KM expansion, identify the coefficients in terms of the known quantities and, finally, explicitly compute the first coefficients.

Let me write the terms inside the sum in Eq.(2.12) and use the Taylor expansion in Eq.(2.13) to isolate the terms without derivatives.

$$\begin{aligned} &\rho(\mathbf{x}+\boldsymbol{\delta}, t) T^{-\boldsymbol{\delta}}(\mathbf{x}+\boldsymbol{\delta}) - \rho(\mathbf{x}, t) T^{\boldsymbol{\delta}}(\mathbf{x}) = \\ &\qquad = \rho(\mathbf{x}, t) T^{-\boldsymbol{\delta}}(\mathbf{x}) - \rho(\mathbf{x}, t) T^{\boldsymbol{\delta}}(\mathbf{x}) + \\ &\qquad + \sum_{n=1}^{+\infty} \frac{1}{n!} \sum_{i_1,\ldots,i_n}^{s} \left[\prod_{m=1}^{n} \delta_{i_m} \frac{\partial}{\partial x_{i_m}}\right] \rho T^{-\boldsymbol{\delta}} \end{aligned} \tag{2.16}$$

I now perform the sum over the first two terms

$$\begin{aligned} &\sum_{\boldsymbol{\delta}\neq 0} \left[\rho(\mathbf{x}, t) T^{-\boldsymbol{\delta}}(\mathbf{x}) - \rho(\mathbf{x}, t) T^{\boldsymbol{\delta}}(\mathbf{x})\right] = \\ &\qquad = \rho(\mathbf{x}, t) \sum_{\boldsymbol{\delta}\neq 0} \left[T^{-\boldsymbol{\delta}}(\mathbf{x}) - T^{\boldsymbol{\delta}}(\mathbf{x})\right] \\ &\qquad = \rho(\mathbf{x}, t) \left(\sum_{\boldsymbol{\delta}\neq 0} T^{-\boldsymbol{\delta}}(\mathbf{x}) - \sum_{\boldsymbol{\delta}\neq 0} T^{\boldsymbol{\delta}}(\mathbf{x})\right) \\ &\qquad = 0 \end{aligned} \tag{2.17}$$

Then, I sum the remaining terms, reorder them and identify the KM coefficients $D^{(n)}(\mathbf{x})$.

$$
\begin{aligned}
\sum_{\boldsymbol{\delta}\neq 0}\sum_{n=1}^{+\infty}\frac{1}{n!}\sum_{i_1,\dots,i_n}^{s}\left[\prod_{m=1}^{n}\delta_{i_m}\frac{\partial}{\partial x_{i_m}}\right]\rho(\mathbf{x})T^{-\boldsymbol{\delta}}(\mathbf{x}) = \\
&= \sum_{n=1}^{+\infty}(-1)^n\sum_{i_1,\dots,i_n}^{s}\sum_{\boldsymbol{\delta}\neq 0}\left[\prod_{m=1}^{n}\frac{\partial}{\partial x_{i_m}}\delta_{i_m}\right]\frac{(-1)^n}{n!}T^{-\boldsymbol{\delta}}(\mathbf{x})\rho(\mathbf{x},t) \\
&= \sum_{n=1}^{+\infty}(-1)^n\sum_{i_1,\dots,i_n}^{s}\sum_{\boldsymbol{\delta}\neq 0}\left[\prod_{l=1}^{n}\frac{\partial}{\partial x_{i_l}}\right]\left[\prod_{m=1}^{n}\delta_{i_m}\right]\frac{(-1)^n}{n!}T^{-\boldsymbol{\delta}}(\mathbf{x})\rho(\mathbf{x},t) \\
&= \sum_{n=1}^{+\infty}(-1)^n\sum_{i_1,\dots,i_n}^{s}\left[\prod_{l=1}^{n}\frac{\partial}{\partial x_{i_l}}\right]\left(\frac{(-1)^n}{n!}\sum_{\boldsymbol{\delta}\neq 0}\left[\prod_{m=1}^{n}\delta_{i_m}\right]T^{-\boldsymbol{\delta}}(\mathbf{x})\right)\rho(\mathbf{x},t) \\
&= 1/N\sum_{n=1}^{+\infty}(-1)^n\sum_{i_1,\dots,i_n}^{s}\left[\prod_{l=1}^{n}\frac{\partial}{\partial x_{i_l}}\right]D^{(n)}_{i_1,\dots,i_n}(\mathbf{x})\rho(\mathbf{x},t) \qquad (2.18)
\end{aligned}
$$

One gets for the KM coefficients Eq.(2.19). This can be used to calculate the Drift, $D^{(1)}_{k}(\mathbf{x})$, and Diffusion, $D^{(2)}_{kl}(\mathbf{x})$. Notice that the n-th KM coefficient contains a factor which is roughly $N\,|\boldsymbol{\delta}|^n$, which proves that the terms in this KM expansion are increasingly small. In addition, it proves that the dynamics of infinite populations is deterministic since all coefficients but the first tend to zero and, therefore, Langevin equation becomes an ordinary differential equation.

$$
D^{(n)}_{i_1,\dots,i_n}(\mathbf{x}) = N\frac{(-1)^n}{n!}\sum_{\boldsymbol{\delta}\neq 0}\left[\prod_{m=1}^{n}\delta_{i_m}\right]T^{-\boldsymbol{\delta}}(\mathbf{x}) \qquad (2.19)
$$

2.3 One-Step Processes

Populations evolve when individuals change their strategy. When an individual with a given strategy decides to change, the number of individuals with this strategy is reduced by one and the strategy for which he changes to gains a new member. This means one has a *birth-death* or *one step process*, keeping the total number of elements. The underlying assumption when using this kind of processes is that the probability that two different individuals change strategy in a time interval τ is $\mathcal{O}\left(\tau^2\right)$ [44], making Def.3 appropriated.

When referring to the process, I will eventually introduce the extra strategy index just to keep track of what I am doing; symbolically, $\mathbf{i} = \{i_1, ..., i_s\} = (i_1, ..., i_s, i_{s+1} = N - i_1 - ... - i_s)$. Then, if one considers

Definition 3 One Step Process

The process $\mathbf{i}$ is a *one step process* if its transition probability per unit time between states $\mathbf{i}$ and $\mathbf{i}'$, $T_{\mathbf{i}\mathbf{i}'}$, is zero for all non adjacent configurations.

all $s+1$ strategies, the configuration of strategies at a given time is $\mathbf{i} = (i_1, ..., i_s, i_{s+1})$ and it can only move to a configuration $\mathbf{i}' = (i'_1, ..., i'_s, i'_{s+1}) = (i_1+\Delta_1, ..., i_{s+1}+\Delta_{s+1})$, where, either all Δ_k are null, or only two of them are non-zero and, respectively, 1 and -1, which identifies the adjacent states. The probability that the system changes into states $\mathbf{i}'$ that do not obey these conditions is zero, $T_{\mathbf{i}'\mathbf{i}} = 0$. When all $\Delta_k = 0$, the system remains unchanged, $\mathbf{i}' = \mathbf{i}$, and the transition probability correspondent to this event can be calculated from the remaining as $T_{\mathbf{i}\mathbf{i}} = 1 - \sum_{\mathbf{i}'\neq\mathbf{i}} T_{\mathbf{i}'\mathbf{i}}$. Hence, the determination of the transition probabilities between adjacent states, allows one to solve the problem. This will be left for Chapter 3.

In this section, I will present and derive several general results related to these processes. Then, I will restrict the analysis to populations with only two strategies so I can introduce concepts as *fixation probability* and *fixation time* and motivate the study of stationary distributions.

2.3.1 Drift and Diffusion

Let me start by explicitly computing the Drift and Diffusion coefficients for this kind of processes. Now that I am considering birth-death processes, notice that the possible transition directions, $\boldsymbol{\delta}$, are under the assumptions of the derivation since all its entries are null except two of them which are, respectively, $1/N$ and $-1/N$, see Sec.2.2. Their explicit form is: $\boldsymbol{\delta} = \{\ldots, \delta_k, \ldots, \delta_l, \ldots\} = \{0, \ldots, 0, \pm 1/N, 0, \ldots\ 0, \mp 1/N, 0, \ldots, 0\}$. Using Eq.(2.19) one writes

$$\begin{aligned}
D_k^{(1)}(\mathbf{x}) &= -N \sum_{\boldsymbol{\delta}\neq 0} \delta_k T^{-\boldsymbol{\delta}}(\mathbf{x}) \\
&= -N \sum_{\boldsymbol{\delta}:\delta_k\neq 0} \delta_k T^{-\boldsymbol{\delta}}(\mathbf{x}) \\
&= -\frac{N}{N}\left(\sum_{\boldsymbol{\delta}:\delta_k=1/N} T^{-\boldsymbol{\delta}}(\mathbf{x}) - \sum_{\boldsymbol{\delta}:\delta_k=-1/N} T^{-\boldsymbol{\delta}}(\mathbf{x})\right) \\
&= -\left(\sum_{\boldsymbol{\delta}:\delta_k=1/N} T^{-\boldsymbol{\delta}}(\mathbf{x}) - \sum_{\boldsymbol{\delta}:\delta_k=1/N} T^{\boldsymbol{\delta}}(\mathbf{x})\right) \\
&\equiv \left(T^{S_k+}(\mathbf{x}) - T^{S_k-}(\mathbf{x})\right)
\end{aligned} \tag{2.20}$$

and

$$
\begin{aligned}
D^{(2)}_{kl}(\mathbf{x}) &= \frac{N}{2} \sum_{\boldsymbol{\delta} \neq 0} \delta_k \delta_l T^{-\boldsymbol{\delta}}(\mathbf{x}) \\
&= \frac{N}{2} \sum_{\boldsymbol{\delta}: \delta_k \neq 0 \wedge \delta_l \neq 0} \delta_k \delta_l T^{-\boldsymbol{\delta}}(\mathbf{x})
\end{aligned} \tag{2.21}
$$

Here is convenient to separate two distinct cases: $k = l$ and $k \neq l$. In the former case one gets Eq.(2.22) while, in the latter, one gets Eq.(2.23).

$$
\begin{aligned}
D^{(2)}_{kk}(\mathbf{x}) &= \frac{N}{2} \sum_{\boldsymbol{\delta}: \delta_k \neq 0} \delta_k \delta_k T^{-\boldsymbol{\delta}}(\mathbf{x}) \\
&= \frac{N}{2N^2} \left(\sum_{\boldsymbol{\delta}: \delta_k = 1/N} T^{-\boldsymbol{\delta}}(\mathbf{x}) + \sum_{\boldsymbol{\delta}: \delta_k = -1/N} T^{-\boldsymbol{\delta}}(\mathbf{x}) \right) \\
&= \frac{1}{2N} \left(\sum_{\boldsymbol{\delta}: \delta_k = 1/N} T^{-\boldsymbol{\delta}}(\mathbf{x}) + \sum_{\boldsymbol{\delta}: \delta_k = 1/N} T^{\boldsymbol{\delta}}(\mathbf{x}) \right) \\
&\equiv \frac{1}{2N} \left(T^{S_k -}(\mathbf{x}) + T^{S_k +}(\mathbf{x}) \right)
\end{aligned} \tag{2.22}
$$

$$
\begin{aligned}
D^{(2)}_{kl}(\mathbf{x}) &= \frac{1}{2N} \left(-T^{-\{\ldots,\delta_k=1,\ldots,\delta_l=-1,\ldots\}}(\mathbf{x}) - T^{-\{\ldots,\delta_k=-1,\ldots,\delta_l=1,\ldots\}}(\mathbf{x}) \right) \\
&= -\frac{1}{2N} \left(T^{\{\ldots,\delta_k=-1,\ldots,\delta_l=1,\ldots\}}(\mathbf{x}) + T^{\{\ldots,\delta_k=1,\ldots,\delta_l=-1,\ldots\}}(\mathbf{x}) \right) \\
&\equiv -\frac{1}{2N} \left(T_{S_k \to S_l}(\mathbf{x}) + T_{S_l \to S_k}(\mathbf{x}) \right)
\end{aligned} \tag{2.23}
$$

Where in the last equalities I have used the definitions (2.24) and (2.25). $T^{S_k \pm}(\mathbf{x})$ is the sum of all transition probabilities that increase or decrease the strategy k, respectively, and $T_{S_k \to S_l}(\mathbf{x})$ is the probability that an individual with strategy k changes into a strategy l.

$$
T^{S_k \pm}(\mathbf{x}) := \sum_{\boldsymbol{\delta}: \delta_k = 1/N} T^{\pm \boldsymbol{\delta}}(\mathbf{x}) \tag{2.24}
$$

$$
T_{S_k \to S_l}(\mathbf{x}) := T^{\{\ldots,\delta_k=-1,\ldots,\delta_l=1,\ldots\}}(\mathbf{x}) \tag{2.25}
$$

I finally have Drift and Diffusion written in terms of quantities that can be computed even in the finite system. I can finally write a Langevin equation, which has now a very intuitive interpretation, Eq.(2.26). The deterministic trend and the most probable direction in phase space, given by the Drift, are a balance between the probability of increasing and decreasing a given strategy. The dispersion of the fraction of individuals with a given strategy across the configurations in the phase space decreases with the increase of population and the major terms in the Diffusion are a sum of the transitions in opposite directions. In the unidimentional case this is particularly easy to notice, Eq.(2.27).

$$\frac{d\mathbf{x}}{dt} = \boldsymbol{T}^{+}(\mathbf{x}) - \boldsymbol{T}^{-}(\mathbf{x}) + \frac{1}{\sqrt{N}}\sqrt{N\boldsymbol{D}^{(\mathbf{2})}}\boldsymbol{\Gamma}. \tag{2.26}$$

$$\frac{dx}{dt} = T^{+}(x) - T^{-}(x) + \frac{1}{\sqrt{N}}\sqrt{\frac{T^{+}(x)+T^{-}(x)}{2}}\,\Gamma. \tag{2.27}$$

In any case, the study of the Drift and Diffusion will be the key for the comprehension of the population dynamics. In what follows I will discuss briefly how to interpret these functions and how to relate them to the population dynamics.

For large enough populations the fluctuations in the individuals of a given strategy will be small compared to the global trend. Therefore, neglecting the stochastic term in the Langevin equation, one finds a system of ordinary differential equation, called the replicator equation in population modeling: $\frac{d\mathbf{x}}{dt} = \mathbf{D}^{(\mathbf{1})}(\mathbf{x})$, where $\mathbf{D}^{(\mathbf{1})}(\mathbf{x}) = \mathbf{T}^{+}(\mathbf{x}) - \mathbf{T}^{-}(\mathbf{x})$. This is an intuitive way to motivate the Drift as the central direction in phase space. Formally, for prediction of the system's evolution, in general, short time propagators need to be taken into account, $P_{1|1}(\mathbf{x}, t+\tau|\mathbf{x}', t)$, which involve both the stochastic (Diffusion) and the deterministic (Drift) parts of the dynamics [45]; computing the gradient in $\mathbf{x}$ for the each point $\mathbf{x}'$ one finds $\boldsymbol{D}^{(\mathbf{1})}(\mathbf{x}')$ as the most probable direction.

Let me get back to the finite systems. To make this transition clear, I will use the $\mathbf{i}$ variables whenever I am in the finite system and the $\mathbf{x}$ variables when I am talking about the infinite limit. The functional dependence distinction is done indexing the variable whenever I am talking about the discrete system's functions.

If the elements in the population can only adopt two different strategies, say S_1 and S_2, I have a one-dimensional problem. Let i_1 be the number of individuals with a given strategy S_1 (then $N - i_1$ is the number of the remaining strategy) and our configuration space be represented as a set of points confined in a line segment such that $0 \leq i_1 \leq N$, see Eq.(2.3). Whenever $D^{(1)}_{i_1} > 0$ (< 0) the population will tend to see the number of elements with strategy S_1 increase (decrease). Considering infinite populations, if $D^{(1)}(x_1) = 0$ one finds a fixed point which can be stable or unstable, as in traditional dynamical systems. However, since populations are finite, stochasticity is present there is no fixation and these fixed points analogues act as attractors or repellers, respectively. Diffusion is also a function of i_1 and is always non-negative, so it can be plotted as the Drift if one is interested in it. Fig.2.1a shows an example of simple way of gathering all this information.

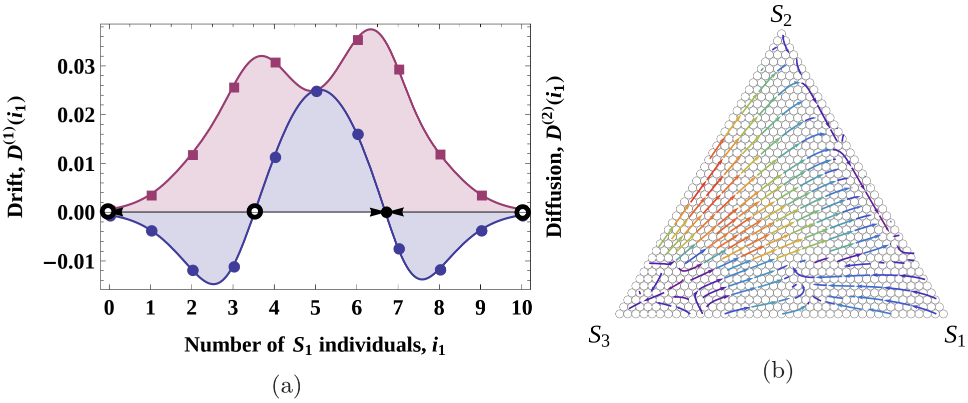


Figure 2.1: Drift and Diffusion representation
(a) Illustration of Drift and Diffusion representation for a population whose individuals can opt between two different strategies. The line with circles represents the Drift and the line with the squares represents the Diffusion. When the Drift is positive (negative) an arrow pointing right (left) is placed in the configuration axis, indicating the general tendency of the population. The dots in this axis represent fixed points analogues: a filled dot represents a fixed point which acts as an attractor and an empty dot acts as a repeller. Notice that those fixed points do not necessarily belong to the configuration state space, for it is discrete, and that Diffusion in those points is necessarily non-zero (see, eg. Eq.(2.27)). (b) Illustration of Drift representation for a population whose individuals can opt between three different strategies. The vector field indicates the direction of preferential motion and a color scale is used to indicate its strength. Fixed point analogues are also visible in this image though they are not represented to avoid overloading the image. I will be using the background configuration space to plot the stationary distribution (see next subsection). Diffusion is not represented for the same reason.

However, if the elements in the population can adopt more than two different strategies, the representation can get a little bit tricky. Notice that, in general, the Drift is a vector field but the Diffusion is a tensor field that is represented by a matrix. Therefore, using its eigenvectors and making their magnitude equal to the correspondent eigenvalue, one is able plot the Diffusion matrix [48, 47].

In this thesis, I will also require the representation of three different strategies. Therefore, consider a population whose individuals can opt for the strategies S_1, S_2 or S_3. Let $\mathbf{i} = \{i_1, i_2\}$ denote the possible configurations, or, $\mathbf{i} = (i_1, i_2, i_3 = 1-i_1-i_2)$. These configurations can be represented in a discrete simplex defined by the domain in equation Eq.(2.3). Because I am dealing with one-step processes, only one of the individual can change strategy at time and, thereafter, there are six possible transitions. The way to allow this transitions graphically, such that they are all equivalent, is to arrange the population in a hexagonal lattice. Fig.2.1b represents some Drift vector field and Fig.2.2 the detail of the hexagonal arrangement.

2.3.2 A geometric interpretation of the Drift

In this subsection, I will introduce a geometric interpretation for the Drift. Fig.2.2 contains a local representation of a configuration and its possible transitions. To every nearest neighbor, $\mathbf{i}'$, I associate a vector with magnitude $T_{\mathbf{i}'\mathbf{i}}$ with the direction of $\mathbf{i}' - \mathbf{i}$. Then I choose the standard non-orthogonal basis, the unitary vectors $\mathbf{u_1}$ and $\mathbf{u_2}$ with the direction of $\frac{\partial \mathbf{i}}{\partial i_1}$ and $\frac{\partial \mathbf{i}}{\partial i_2}$, respectively. Performing the sum of these vectors one finds a new local vector, $\mathbf{g_i}$, the gradient of selection, which contains information about all possible transitions and, using Eq.(2.24), can be written in the form of Eq.(2.28), the Drift.

$$\boldsymbol{g}_{\mathbf{i}} = (T_{\mathbf{i}}^{S_1+} - T_{\mathbf{i}}^{S_1-})\mathbf{u_1} + (T_{\mathbf{i}}^{S_2+} - T_{\mathbf{i}}^{S_2-})\mathbf{u_2} \tag{2.28}$$

The entries of the gradient of selection correspond to the Drift in each direction and, therefore, $\mathbf{g_i} = \mathbf{D}_{\mathbf{i}}^{(\mathbf{1})}$.

2.3.3 Fixation Problem and Stationary Distributions

In this subsection I will consider one-dimensional processes, i.e. populations whose individuals can opt for two different strategies: S_1 and S_2. This way, $\mathbf{i} = i_1 = i$.

Suppose the population dynamics has some stable and unstable fixed points analogues, some $\{x \in \Re : D_{xN}^{(1)} = 0\}$: attractors and repellers. The population will, therefore, spend most of its time around the attractors and little time near repellers but, since it is finite, stochasticity will allow all configurations to be explored. Eventually, the configuration in which the whole population adopts strategy S_1 or S_2 will occur and the imitation process, which is the motive of the dynamics, will end: an individual can only imitate his own strategy. Essentially, if only the imitation process is considered, evolutionary dynamics in finite populations will (only) stop whenever the population reaches a monomorphic state [49, 40].

Hence, in addition to the analysis of the shape of $\mathbf{g_i} = g_i$, the gradient of selection, or Drift, often one of the quantities of interest in studying the evolutionary dynamics in finite populations is the probability ϕ_i that the system fixates in a monomorphic S_1 state, starting from, for instance, a given number i of S_1's.

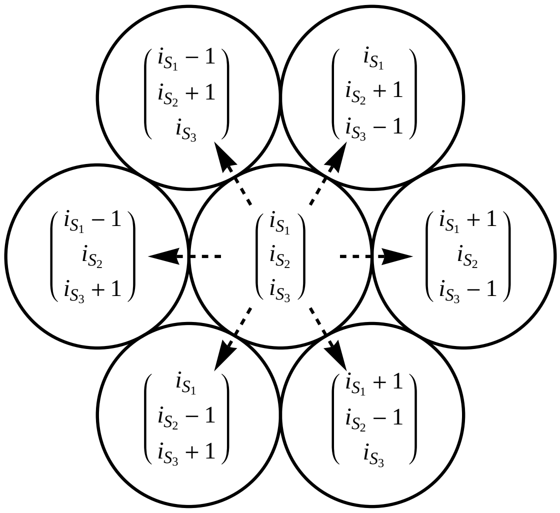


Figure 2.2: Phase space representation
Local representation of the phase space and possible transitions for a bi-dimensional one-step process. A vector can be associated with every transition between the element **i** and each adjacent element. The sum of these vectors corresponds to the Drift in the configuration **i**, which is often called the gradient of selection, $\mathbf{g_i}$ (see Chapter 3).

The *fixation probability* of i S_1's, ϕ_i, depends on the ratio $\lambda_i = T_i^-/T_i^+$, being given by [50]

$$\phi_i = \sum_{l=0}^{i-1} \prod_{l'=1}^{l} \lambda_{l'} / \sum_{l=0}^{N-1} \prod_{l'=1}^{l} \lambda_{l'} \tag{2.29}$$

Under neutral selection (that is, when the probability of changing or not changing strategy is 1/2), the fixation probability trivially reads $\phi_i^0 = i/N$, providing a convenient reference point [20, 50, 40, 51]. For a given i, whenever $\phi_i > \phi_i^0$, natural selection will favor cooperative behavior, the opposite being true when $\phi_i < \phi_i^0$.

Yet, even if fixation in one of the two absorbing states is certain ($i = 0$ and $i = N$), the time required to reach it can be arbitrarily long. This is particularly relevant in the presence of those basins of attraction with polymorphic stable configurations, which correspond to finite population analogues of co-existence equilibria in infinite populations. For instance, in large populations, the time required for fixation, t_i, can increase significantly. Following Antal and Scheuring [52], the average number of updates t_i the population takes to reach monomorphic S_1 , starting from i individuals with strategy S_1, can be written as [52, 53]

$$t_i = -t_1 \frac{\phi_1}{\phi_i} \sum_{l=i}^{N-1} \prod_{l''=1}^{l} \lambda_{l''} + \sum_{l=i}^{N-1} \sum_{l'=1}^{l} \frac{\phi_{l'}}{T_i^+} \prod_{l''=l'+1}^{l} \lambda_{l''} \tag{2.30}$$

where

$$t_1 = \sum_{l=i}^{N-1} \prod_{l'=1}^{l} \frac{\phi_{l'}}{T_{l'}^+} \prod_{l''=l'+1}^{l} \lambda_{l''} \tag{2.31}$$

This way, one complements the simple study of the fixation probabilities, knowing how long they take to fixate. Notice that if the fixation in one of monomorphic state is certain but the time it takes to fixates is too long the first knowledge is rather irrelevant. For example, the existence of a stable equilibrium may turn the analysis of the fixation probability misleading and, therefore, fixation probabilities may fail to characterize in a reasonable way the evolutionary dynamics under general conditions. Moreover, as I mentioned before, stochastic effects in finite populations can be of different nature, going beyond errors in the imitation process. In addition to social learning by imitation dynamics, one can also consider the so-called mutations: random exploration of strategies or any other reason that leads individuals to change their behavior [41]. Under these circumstances, the population will never fixate in none of the two possible monomorphic states (see Chapter 4).

The proper alternative, which overcomes the drawbacks identified in both ϕ_i and g_i, consists in the analysis of the distributions of the complete Markov as mentioned in the very beginning, Sec. 2.1. In general, for the complete solution of the problem, one would solve M-Equation, Eq.(2.4), to compute the PDF evolution. This can be very time and resource consuming and, furthermore, the analysis of the results would not be simple. However, I am not interested in transient distributions and, thereafter, I can look for stationary solutions of this equation. In general, this is obtained from the eigenvector associated to the eigenvalue 1 of the $T_{\mathbf{i}'\mathbf{i}}$ matrix. Notice that as long as the states are numerable this matrix is always buildable for an arbitrary number of strategies.

To finish this subsection I will derive a solution that avoids solving this eigenproblem for one-dimensional one-step processes. For an alternate derivation see [44]. Let i be a limited one-step process: $i = 0, 1, ..., N$. The transition probability from state i to i' is $T_{i'i}$. Thus, the M-equation for the probability $p_i(t)$ of the configuration i at $t > t_0$, for an initial condition $p_i(t_0) = \delta_{ii_0}$, is [44]:

$$\Delta p_i(t) = \sum_{i'=0}^{N} (T_{ii'} p_{i'}(t) - T_{i'i} p_i(t)) \tag{2.32}$$

$$= \sum_{i'=0, i' \neq i}^{N} (T_{ii'} p_{i'}(t) - p_i(t) T_{i'i}) \tag{2.33}$$

$$= \begin{cases} T_{i\,i-1} p_{i-1}(t) + T_{i\,i+1} p_{i+1}(t) - (T_{i-1\,i} + T_{i+1\,i}) p_i(t), & 0 < i < N \\ T_{01} p_1(t) - T_{10} p_0(t), & i = 0 \\ T_{N\,N-1} p_{N-1}(t) - T_{N-1\,N} p_N(t), & i = N \end{cases}$$

Searching for a stationary solution, $\Delta p_i(t) = 0$ for all i, and one can use the recurrent relation:

$$i = 0 \Rightarrow p_1 = p_0 T_{10} / T_{01} \tag{2.34}$$

$$0 < i < N \Rightarrow p_{i+1} = (p_i(T_{i-1i} + T_{i+1i}) - p_{i-1} T_{ii-1}) / T_{ii+1} \tag{2.35}$$

$$i = N \Rightarrow p_N = p_{N-1} T_{NN-1} / T_{N-1N} \tag{2.36}$$

Which allows one to write all p_i as a function of p_0. However, from Eq.(2.35) and using Eq.(2.34), one has:

$$T_{12} p_2 = p_1(T_{01} + T_{21}) - p_0 T_{10} \tag{2.37}$$

$$= p_1(T_{01} + T_{21}) - p_1 T_0 1 \tag{2.38}$$

$$= p_1 T_{21}. \tag{2.39}$$

Which is the same kind of relation as in Eq.(2.34). Thus, is natural to assume the hypothesis:

$$p_n = p_{n-1} T_{nn-1} / T_{n-1n}, \tag{2.40}$$

Valid for $n = 1$ via Eq.(2.34). Using Eq.(2.35) one writes:

$$T_{nn+1} p_{n+1} = p_n(T_{n-1n} + T_{n+1n}) - p_{n-1} T_{nn-1} \tag{2.41}$$

$$\overset{hip}{=} p_n(T_{n-1n} + T_{n+1n}) - p_n T_{n-1n} \tag{2.42}$$

$$= p_n T_{n+1n} \tag{2.43}$$

Which validates the hypothesis, by induction, for $i \geq 1$. Recurrently one has:

$$p_i = p_{i-1} T_{ii-1} / T_{i-1i} = p_{i-2} T_{i-1i-2} / T_{i-2i-1} T_{ii-1} / T_{i-1i} = \ldots =$$

$$= p_0 \frac{T_{10}}{T_{01}} \frac{T_{21}}{T_{12}} \cdots \frac{T_{i-1i-2}}{T_{i-2i-1}} \frac{T_{ii-1}}{T_{i-1i}} \tag{2.44}$$

Finally, each p_i is determined as a function of the ratio between $T_l^+ = T_{l+1l}$ and $T_{l+1}^- = T_{ll+1}$, R_l, and p_0 determined by normalization:

$$p_i = p_0 \prod_{l=0}^{i-1} \frac{T_{l+1l}}{T_{ll+1}} \equiv p_0 \prod_{l=0}^{i-1} R_l,\ \ p_0 = \left(\sum_{i=0}^{N} \prod_{l=0}^{i-1} R_l \right)^{-1} \tag{2.45}$$

Chapter 3

Evolutionary Game Theory

So far, I have reduced the problem of building up a model that describes the evolution of a population to the ability of writing some transition probabilities. To do this, one must understand what makes individuals to change their own strategy. As already mentioned, an imitation principle must be behind the evolution of the population but this is not enough, for it systematically leads the population to homogeneous states.

Individuals tend to copy others whenever these appear to be more successful. Contrary to strategies defined by a contingency plan, which, as some argue [54], are unlikely to be maintained for a long time scale, this social learning (or evolutionary) approach allows policies to change as time goes by [24, 36, 55]. Likely, these policies will be influenced by the behavior (and achievements) of others, as it happens in the context of donations to public goods [37, 38, 39]. This also takes into account the fact that agreements may be vulnerable to renegotiation, as individuals may agree on intermediate goals or assess actual and future consequences of their choices to revise their position [1, 2, 8, 56, 57, 58].

Moreover, one must also consider random exploration of strategies or any other reason that leads individuals to change their behavior, *mutation*. In the simplest scenario, this creates a modified set of transition probabilities, with an additional random factor encoding the probability of a mutation, μ, in each update step. Under these circumstances, the population will never fixate in none of the two possible monomorphic states and will evolve preferentially according the imitation process.

To this point, the missing transition probabilities correspond to the probability that an element with a given strategy, S_l, changes into another specific strategy, S_k. Evidently this can depend on the couple of strategies but I will assume a common functional form for all strategies; the process of decision has the same rule. The rule I use to compute $T_{S_l \to S_k} \equiv T_{\{\ldots,i_k+1,\ldots,i_l-1,\ldots\}\{\ldots,i_k,\ldots,i_l,\ldots\}}$ is the *Fermi update rule with mu-*

tation, or *pairwise comparison rule* [40]:

- Considering all individuals are equally likely to conceive the change of strategy, the probability an individual with strategy S_l gives it a try is i_l/N.

- If the individual compares to all others in the same way, like for well-mixed populations, he compares his strategy to the strategy S_k with probability $i_k/(N-1)$.

- In the comparison process, the individual more likely changes strategy if his strategy is worse than the one he is comparing to. This is accomplished using a Fermi distribution, $(1+\exp(\beta\Delta_{S_lS_k}))^{-1}$, which introduces errors in the imitation process, where β represents the intensity of this selection and $\Delta_{S_lS_k}$ quantifies how better is strategy S_l compared to S_k. For $\beta << 1$, selection is weak and its effect is but a small perturbation to random drift in behavioral space.

- Additionally, one may introduce a parameter μ, the *mutation*, that allows transitions between strategies independent of how *good* they are. When μ has its maximum value, 1, the individual changes (or not) to any strategy with equal probability.

These four ingredients build Eq.(3.1) [40, 59, 41].

$$T_{S_l\to S_k} = \frac{i_l}{N}\left(\frac{i_k}{N-1}\frac{1-\mu}{1+\exp(\beta\Delta_{S_lS_k})} + \frac{\mu}{s}\right) \tag{3.1}$$

This formulation allows one to explicitly compute the Drift and Diffusion for a Fermi update rule using Eq.(2.20) and Eq.(2.21), see Sec. 3.1.

3.1 Drift and Diffusion for Fermi Update Rule

In this section I will calculate all Drift and Diffusion using the Fermi update rule. I use the finite system notation.

Consider the quantities $T_{\mathbf{i}}^{S_k+} \pm T_{\mathbf{i}}^{S_k-}$, whose definitions are given, in Eq.(2.24), in terms of $T_{S_l\to S_k}$, Eq.(3.1). Then,

$$
\begin{aligned}
T_{\mathbf{i}}^{S_k+} \pm T_{\mathbf{i}}^{S_k-} = \sum_{l\neq k}^{s+1} & \left[\frac{i_l}{N}\left(\frac{i_k}{N-1}\frac{1-\mu}{1+\exp(\beta\Delta_{S_lS_k})}+\frac{\mu}{s}\right)\pm\right. \\
& \left.\pm\frac{i_k}{N}\left(\frac{i_l}{N-1}\frac{1-\mu}{1+\exp(\beta\Delta_{S_kS_l})}+\frac{\mu}{s}\right)\right] \qquad (3.2) \\
= \sum_{l\neq k}^{s+1} & \left[\frac{i_k(1-\mu)}{N(N-1)}i_l\left(\frac{1}{1+\exp(\beta\Delta_{S_lS_k})}\pm\right.\right. \\
& \left.\left.\pm\frac{1}{1+\exp(\beta\Delta_{S_kS_l})}\right)+\frac{\mu}{sN}(i_l\pm i_k)\right] \qquad (3.3)
\end{aligned}
$$

Using the conceptual symmetry $\Delta_{S_lS_k} = -\Delta_{S_kS_l}$ and the identities $(1+\exp(x))^{-1} - (1+\exp(-x))^{-1} \equiv \tanh(-x/2)$ and $(1+\exp(x))^{-1} + (1+\exp(-x))^{-1} \equiv 1$ I can write Eqs.(3.4).

$$
T_{\mathbf{i}}^{S_k+} - T_{\mathbf{i}}^{S_k-} = \frac{i_k(1-\mu)}{N(N-1)}\sum_{l\neq k}^{s+1}\left[i_l\tanh\left(\frac{\Delta_{S_kS_l}}{2}\right)\right]+\frac{\mu}{sN}(N-(s+1)i_k) \tag{3.4a}
$$

$$
T_{\mathbf{i}}^{S_k+} + T_{\mathbf{i}}^{S_k-} = \frac{i_k(1-\mu)}{N(N-1)}(N-i_k)+\frac{\mu}{sN}(N+(s-1)i_k) \tag{3.4b}
$$

In the same way, one computes

$$
T_{S_k\to S_l} + T_{S_l\to S_k} = \frac{i_k i_l(1-\mu)}{N(N-1)}+\frac{\mu}{sN}(i_k+i_l) \tag{3.5}
$$

$T_{\mathbf{i}}^{S_k+} - T_{\mathbf{i}}^{S_k-}$ is the Drift vector and the remaining functions are part of the Diffusion matrix.

Notice that $T_{\mathbf{i}}^{S_k+} + T_{\mathbf{i}}^{S_k-}$ and $T_{S_k\to S_l} + T_{S_l\to S_k}$ are independent of the strategies. These elements build the diffusion matrix and, therefore, Diffusion is the same for all conceivable games and strategies once one chooses the Fermi update rule. If one is interested in the study of fluctuations of strategies in populations, the mutation should follow some more complex rule, otherwise no contributions from the games themselves will arise.

3.2 Interaction

By now, I only need to define how subjects compare their strategies. I have already mentioned "*better* strategies" but, so far, all I did was packing this information in $\Delta_{S_lS_k}$. Actually, $\Delta_{S_lS_k}$ is indexed to a given configuration too, since a strategy is only said to be good in a given situation. I will consider that each strategy, S_k, for each configuration, has a well defined fitness, $f_{S_k\mathbf{i}}$. The bigger its fitness, the better succeeded is the strategy. In this sense, one writes $\Delta_{S_lS_k} = f_{S_l} - f_{S_k}$. The way the whole procedure is built supposes this fitness must be accessible to the individuals since it is part of their decision process. Evidently, this can only be done via interaction with the other players.

I will consider n-person interactions: an individual interacts with all N players but in groups of size n. In these small games one considers that each player has a payoff, depending on the strategy he uses in the game. This payoff is represented by a numerical amount, which is positive whenever the interaction benefits the individuals and negative when they are harmed. Since individuals with the same strategy behave in the same way, the payoff is a characteristic of the strategy and, ultimately, the payoff, P_{S_k}, is what defines the strategy.

So that individuals are allowed to make good decisions, many of this interaction will take part between each decision process. Thus, a mean field approximation is considered and an average payoff, representing the social success, must be taken into account.

Definition 4 Fitness

The *fitness* of a strategy S_l, $f_{S_l\mathbf{i}}$, is the average payoff a single individual with strategy S_l obtains over all possible games that can be played in a given configuration of the whole population.

For finite, well-mixed populations of size N, this average is accomplished using an hypergeometric sampling, sampling without replacement [60]. Let $\mathbf{j} = \{j_1, ..., j_s\} = (j_1, ..., j_{s+1})$ be the configuration of players in the group of size n with identical definition to $\mathbf{i}$ but replacing N for n, i.e., the configuration of the small group. Then, using Def.4, $f_{S_k\mathbf{i}}$ is given by Eq.(3.6).

$$f_{S_k\mathbf{i}} = \binom{N-1}{n-1}^{-1} \sum_{\substack{j_1,...,j_s \\ j_1+...+j_s \leq n-1}} P_{S_k\mathbf{j}} \binom{i_k - 1}{j_k} \prod_{l\neq k}^{s+1} \binom{i_l}{j_l} \tag{3.6}$$

Where $P_{S_k\mathbf{j}}$ is the payoff of an individual with strategy S_l in an n-person game with configuration $\mathbf{j}$, which must be such that it contains at least one individual with strategy S_k.

If one assumes an infinite population, in a configuration $\mathbf{x} = \{x_1, ..., x_s\} = (x_1, ..., x_{s+1} = 1 - x_1 - ... - x_s)$, where also every individual can potentially interact with everyone else, the fitness of each individual can be obtained from a random sampling of groups. The latter leads to groups whose composition follows a binomial distribution. Hence, for groups of size n and configuration $\mathbf{j}$, I may write the fitness of a given strategy $f_{S_k}(\mathbf{x})$ using Eq.(3.7) [31, 32, 61].

$$f_{S_k}(\mathbf{x}) = \sum_{\substack{j_1,\ldots,j_s \\ j_1+\ldots+j_s\leq n-1}} \frac{(n-1)!}{j_1!\ldots j_{s+1}!} P_{S_k\mathbf{j}} \prod_{l}^{s+1} x_l^{j_l} \tag{3.7}$$

Now on, our mission will be to define proper strategies by giving the behavior of an individual in a group of size n and compare the effects that different strategies have on the behavior of the population.

Chapter 4

The effect of Risk

Now that the framework is settled, I will explore in detail the problem exposed in the introduction: the welfare of our planet. Throughout this chapter I will use most of the results introduced and derived before. I will be considering two strategies and, therefore, the results are particularized to a one-dimensional phase space.

One of the most distinctive features of this complex problem, only recently tested and confirmed by means of actual experiments [10], is the role played by the perception of risk that accrues to all actors involved when taking a decision. Indeed, experiments confirm the intuition that the risk of collective failure plays central role in dealing with climate change. Up to now, the role of risk has remained elusive [1, 2, 62]. Additionally, it is also unclear what is the ideal scale or size of the population engaging in climate summits - whether game participants are world citizens, regions or country leaders -, such that the chances of cooperation are maximized. In this chapter I address these two issues in the context of game theory and population dynamics [60].

The conventional public goods game - the so-called n-person Prisoner's Dilemma - involve a group of n individuals, who can be either Cooperators (C) or Defectors (D). C's contribute a *cost* c to the public good, whereas D's refuse to do so. The accumulated contribution is multiplied by an enhancement factor that returns equally shared among all individuals of the group. This implies a collective return that increases linearly with the number of contributors, a situation that contrasts with many real situations in which performing a given task requires the cooperation of a minimum number of individuals of that group [28, 29, 31, 33, 34, 35, 63, 64, 65]. This is the case in international environmental agreements, which demand a minimum number of ratifications to come into practice [1, 2, 66, 12, 56, 57], but examples abound where a minimum number of individuals, which does not necessarily equal the entire group, must simultaneously cooperate before any outcome (or public good) is produced. Furthermore, it is by now clear that the n-person Prisoner's Dilemma fails short to encompass the role of risk, as much as the non-linearity of most collective action problems.

I will address these problems resorting to a simple mathematical model [60], adopting unusual concepts within political and sustainability science research, such as peer influence and evolutionary game theory. As a result I encompass several of the key elements stated before regarding the climate change conundrum in a single dynamical model.

Throughout this chapter I show how small groups under high risk and stringent requirements toward collective success significantly raise the chances of coordinating to save the planet's climate, thus escaping the tragedy of the commons. In other words, global cooperation is dependent on how aware individuals are concerning the risks of collective failure and on the predefined premises needed to accomplish a climate agreement. Moreover, is shown that to achieve stable levels of cooperation, an initial critical mass of cooperators is needed, which will then be seen as role models and foster cooperation. I will start by presenting the model in Sec. 4.1. In Sec. 4.2, I discuss the situation in which evolution is deterministic and proceed in very large populations. In end the Chapter in Sec. 4.3, where I analyze the evolutionary dynamics of the same dilemma in finite populations under errors and behavioral mutations.

4.1 Model

Consider,once again, a large population of size N, in which individuals engage in an n-person dilemma. Here, each individual is able to contribute or not to a common good, i.e. to cooperate or to defect, respectively. Game participants have each an initial endowment, or *benefit*, b. Cooperators (C's) contribute a fraction of their endowment, the *cost*, $c < b$, while defectors (D's) do not contribute. As previously stated, irrespectively of the scale at which agreements are tried, most demand a minimum number of contributors to come into practice. Hence, whenever parties fail to achieve a previously defined minimum of contributions, they may fail to achieve the goals of such agreement (which can also be understood as the benefit b), being this outcome, in the worst possible case, associated with an appalling doomsday scenario. To encompass this feature in the model one requires a minimum collective investment to ensure success: if the group of size n does not contain at least n_{pg} C's (or, equivalently, a collective effort for the public good of $n_{pg}c$), all members will lose their remaining endowments with a probability r, the *risk*; otherwise, everyone will keep whatever they have. Hence, $n_{pg} < n$ represents a coordination threshold [10, 31] necessary to achieve a collective benefit. As a result, the payoff of a C in a group of size n and $i_C = i$ C's (and $i_D = n - i$ D's) can be written as

$$P_{Ci} = b\{\Theta(i - n_{pg}) + (1 - r)(1 - \Theta(i - n_{pg}))\}, \tag{4.1}$$

where $\Theta(x)$ is the Heaviside step function ($\Theta(x < 0) = 0$ and $\Theta(x \geq 0) = 1$). Similarly, the payoff of a C is given by

$$P_{Ci} = P_{Ci} + c. \tag{4.2}$$

The risk r is here introduced as a probability, such that with probability $(1-r)$ the benefit will be collected independently of the number of contributors in a group. This collective-risk dilemma (CRD) represents a simplified version of the game used in the experiments performed by Milinski et al. [10] on the issue of the mitigation of the effects of climate change, a framework that is by no means the standard approach to deal with International Environmental Agreements and other problems of the same kind [1, 2, 56, 57]. The present formalism has the virtue of depicting black on white the importance of risk and its assessment in dealing with climate change, something that Heal et al. [67, 58] have been conjecturing for quite awhile. At the same time, contrary to those experiments [10], this analysis is general and not restricted to a given group size. Additionally, and unlike most treatments [1], this analysis will not rely on individual or collective rationality. Instead, this model relies on evolutionary game theory combined with one-shot Public Goods games, in which errors are allowed. In fact, this model probably includes the key factors in any real setting, such as bounded rational individual behavior, peer-influence and the importance of risk assessment in meeting the goals defined from the outset.

4.2 Evolution of Collective Action in Large Populations

In the framework of evolutionary game theory, the evolution or social learning dynamics of the fraction $x_C = x$ of C's (and $x_D = 1-x$ of D's) in a large population ($N \to \infty$) is governed by the gradient of selection associated with the replicator dynamics [24, 31, 68], Eq.(4.3). This is easily accomplished using the one-dimensional Langevin equation Eq.(2.27), equivalent to the replicator equation [68] (see Sec.2.3.1).

$$\dot{x} = g(x) \equiv (f_C(x) - f_D(x))x(1 - x) \tag{4.3}$$

The replicator dynamics characterizes the behavioral dynamics of the population, where $f_C(f_D)$ is the fitness of C's (D's), associated with the game payoffs. According to the replicator equation, C's (D's) will increase in the population whenever $g(x) > 0$ ($g(x) < 0$). Using Eq.(3.7) one can write the fitness in this situation.

$$f_C(x) = \sum_{j}^{n-1} \binom{n-1}{j} P_{Cj+1} x^j (1-x)^{n-1-j} \tag{4.4}$$

$$f_D(x) = \sum_{j}^{n-1} \binom{n-1}{j} P_{Dj} x^j (1-x)^{n-1-j} \tag{4.5}$$

where P_{Cj} and P_{Dj} are respectively defined in Eqs.(4.1,4.2).

Fig.4.1 shows that, in the absence of risk, $g(x)$ is always negative. Risk, in turn, leads to the emergence of two mixed internal equilibria, rendering cooperation viable: for finite risk r, both C's (for $x < x_L$) and D's (for $x > x_R$) become disadvantageous when rare. Co-existence between C's and D's becomes stable at a fraction x_R, which increases with r. Collective coordination becomes easier to achieve under high-risk and, once the coordination barrier (x_L) is overcome, high levels of cooperation will be reached.

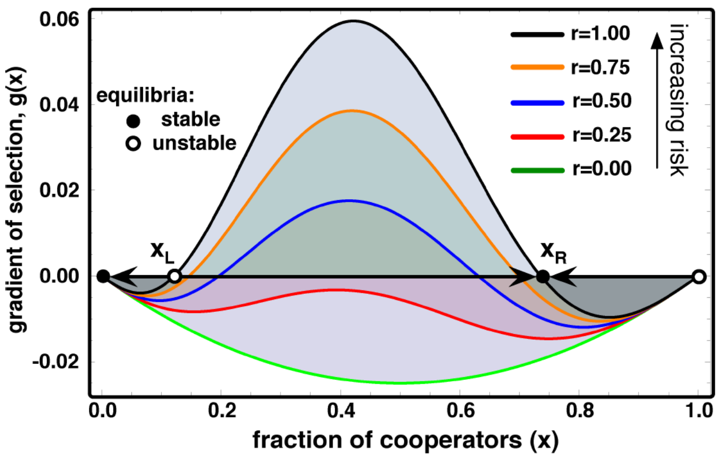


Figure 4.1: Effect of risk – appearance of internal equilibria
For each fraction of C's, if the gradient $g(x)$ is positive (negative) the fraction of C's will increase (decrease). Increasing risk, r, modifies the population dynamics rendering cooperation viable depending on the initial fraction of C's ($n = 6$, $n_{pg} = 3$ and $c/b = 0.1$). The five curves correspond, from top to bottom to, $r = 1, 0.75, 0.5, 0.25, 0$.

As descussed in Chapter 2, the appearance of two internal equilibria under risk can be studied analytically, as the roots of the gradient of selection determines the occurrence of nontrivial equilibria of the replicator dynamics. From the equations above one may write, after some algebra, that

$$g(x) = b \left[\binom{n-1}{n_{pg}-1} x^{n_{pg}-1} (1-x)^{n-n_{pg}} r - c/b \right]. \tag{4.6}$$

Defining the cost-to-risk ratio $\gamma = c/br$, i.e. the ratio between the fraction of the initial budget invested by every C and the risk of losing it, the sign of $g(x)$ is conveniently analyzed by using the polynomial

$$p(x) = \binom{n-1}{n_{pg}-1} x^{n_{pg}-1}(1-x)^{n-n_{pg}} - \gamma \tag{4.7}$$

which, in turn, can be used to determine the critical value $\bar{\gamma}$ below which an interior fixed point $x^* \in (0,1)$ emerges. Indeed, I can prove the following propositions.

Proposition 1. *Let* $\Gamma(x) = \binom{n-1}{n_{pg}-1} x^{n_{pg}-1}(1-x)^{n-n_{pg}}$. *For* $1 < n_{pg} < n$, *there exists a critical cost-to-risk ratio* $\bar{\gamma} = \Gamma(\bar{x}) > 0$ *and fraction of C's* $0 < \bar{x} < 1$ *such that:*

(a) If $\gamma > \bar{\gamma}$, *the evolutionary dynamics has no interior equilibria.*

(b) If $\gamma = \bar{\gamma}$, *then* $\bar{x}$ *is a unique interior equilibrium, and this equilibrium is semistable.*

(c) If $\gamma < \bar{\gamma}$, *there are two interior equilibria* $\{x_L, x_R\}$, *such that* $x_L < \bar{x} < x_R$, x_L, *is unstable and* x_R *stable.*

Proof. Let me start by noticing that

$$\frac{d\Gamma}{dx}(x) = -\binom{n-1}{n_{pg}-1} x^{n_{pg}-2}(1-x)^{n-n_{pg}-1} s(x) \tag{4.8}$$

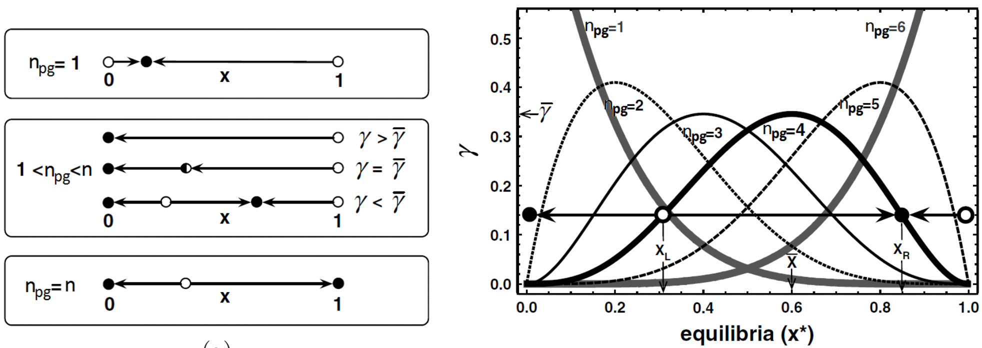
where $s(x) = 1 + (n-1)x - n_{pg}$. Since $n > 2$ and $1 < n_{pg} < n$, then $d\Gamma(x)/dx$ has a single internal root for $\bar{x} = (n_{pg}-1)/(n-1)$. In addition, $s(x)$ is negative (positive) for $x < \bar{x}(x > \bar{x})$, which means that Γ has a global maximum for $x = \bar{x}$. (a) and (b) can now easily follow. Since Γ has a maximum at $\bar{x}$, it follows that $\Gamma(x) = 0$ has no solutions for $\gamma > \bar{\gamma}$ and a single one, at $\bar{x}$, for $\gamma = \bar{\gamma}$. Moreover, both when $x \to 0$ and $x \to 1$, $p(x) < 0$, making $x = 0$ a stable fixed point and $x = 1$ an unstable one. Therefore, if $\bar{x}$ is a root, it must be semistable. To prove (c), I start by noticing that $\Gamma(0) = \Gamma(1) = 0$. From the sign of $s(x)$ (see above), $\Gamma(x)$ is clearly monotonic increasing (decreasing) to the left (right) of $\bar{x}$. Hence, there is a single root $x_L(x_R)$ in the interval $0 < x < \bar{x}$ ($\bar{x} < x < 1$). Since $x = 0$ is stable and $x = 1$ unstable, x_R must be stable and x_L unstable. □

Proposition 2. *For* $n_{pg} = 1$, *if* $\gamma < \bar{\gamma}$, *there is one stable interior equilibrium point in the interval* $0 < x < 1$.

Proof. If $n_{pg} = 1$, $\Gamma(x) = (1-x)^{n-1}$, which is a monotonic decreasing function for $0 < x < 1$. This means that the function $p(x)$ has only one zero in that interval, i.e. there is only one $\bar{x}(0 < \bar{x} < 1)$ such that $p(\bar{x}) = 0$. Given that $p(x)$ is positive (negative) for $x < \bar{x}(x > \bar{x})$ then $\bar{x}$ is a stable equilibrium point. □

Proposition 3. *For $n_{pg} = n$, if $\gamma < \bar{\gamma}$, there is one unstable interior equilibrium point in the interval* $0 < x < 1$.

Proof. If $n_{pg} = n$, $\Gamma(x) = x^{n-1}$, which is a monotonic increasing function for $0 < x < 1$. This means that the function $p(x)$ has only one zero in that interval, i.e. there is only one $\bar{x}(0 < \bar{x} < 1)$ such that $p(\bar{x}) = 0$. Given that $p(x)$ is negative (positive) for $x < \bar{x}(x > \bar{x})$ then $\bar{x}$ is an unstable equilibrium point. □



(a)

(b)

Figure 4.2: Effect of risk – dynamical scenarios for finite populations
(a) Classification of all possible dynamical scenarios when evolving an infinitely large population of C's and D's as a function of γ, n_{pg} and n. A fraction x of an infinitely large population adopts the strategy C; the remaining fraction $1 - x$ adopts D. The replicator equation describes the evolution of x over time. Solid (open) circles represent stable (unstable) equilibria of the evolutionary dynamics; arrows indicate the direction of selection. (b) Internal roots x^* of $g(x)$ for different values of the cost-to-risk ratio $\gamma = c/r$, at fixed group size ($n = 6$) and different coordination thresholds (n_{pg}). For each value of γ one draws a horizontal line; the intersection of this line with each curve gives the value(s) of x^*, defining the internal equilibria of the replicator dynamics. The empty circle represents an unstable fixed point (x_L) and the full circle a stable fixed point (x_R) ($n_{pg} = 4$ and $\gamma = 0.15$ in example).

In Fig.(4.2a), I provide a concise scheme of all possible dynamical scenarios that emerge from collective-risk dilemmas, showing how the coordination threshold and the level of risk play a central role in dictating the viability of cooperation. Fig.(4.2b) also shows the role played by the threshold n_{pg}: for fixed (and low) γ, increasing n will maximize cooperation (increase of x_R) at the expense of making it more difficult to emerge (increase of x_L).

4.3 Evolution of Collective Action in Small Populations

As has been discussed throughout this thesis, real populations are finite and often rather small, contrary to the hypothesis underlying the dynamics portrayed in the previous section. In particular, this is the case of the famous world summits where group and population sizes are comparable and of the order hundreds, as individuals are here associated with nations or their respective leaders. For such population sizes, stochastic effects play an important role and the deterministic description of the previous section may be too simplistic [49]. For finite, well-mixed populations of size N, with $i_C = i$ C's and $i_D = N - i$ D's, the fitness in Eq.(3.6) becomes respectively (see Sec. 3.2)

$$f_{Ci} = \binom{N-1}{n-1}^{-1} \sum_{l=0}^{n-1} \binom{i-1}{l} \binom{N-i}{n-l-1} P_{Cl+1} \tag{4.9}$$

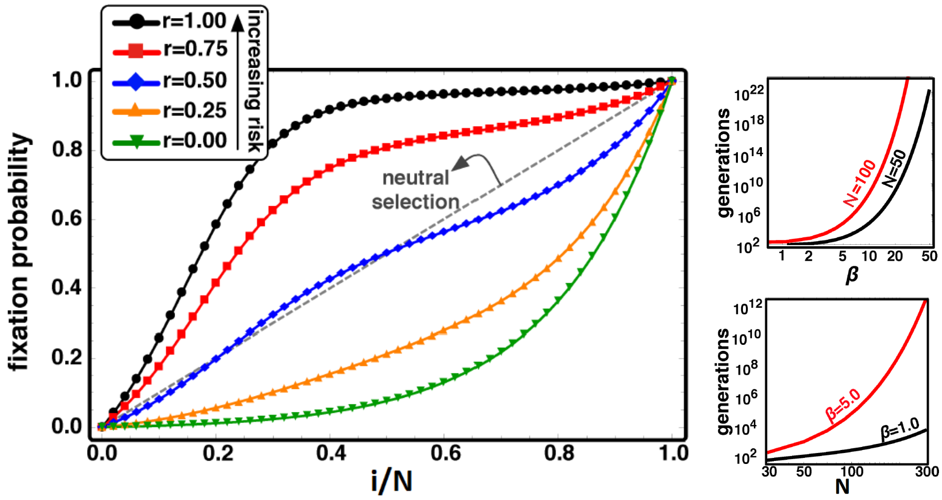
$$f_{Ci} = \binom{N-1}{n-1}^{-1} \sum_{l=0}^{n-1} \binom{i}{l} \binom{N-1-i}{n-l-1} P_{Dl}. \tag{4.10}$$

The gradient of selection, or Drift, is , using Eq.(3.4a), given by

$$g_i = (1-\mu)\frac{i}{N}\left(\frac{N-i}{N-1}\tanh\left(\frac{\beta}{2}(f_{Ci} - f_{Di})\right)\right) + \mu\frac{N-2i}{N} \tag{4.11}$$

When mutation is absent, $\mu = 0$, one can compute quantities introduced before, such as fixation probability and fixation time. In Fig. 4.3 I plot the fixation probability as a function of the initial fraction of C's for different values of risk, and a population of 50 individuals. Even if cooperators remain disadvantageous for a wide range of the discrete frequency of C's (see Fig.4.1), the fixation probability of C's outperforms neutral selection, ϕ_i^0 (picture as a dashed grey line), for most values of i/N. This is due to the stochastic nature of the imitation process, which allows the fixation of rare cooperators, even when they are initially disadvantageous. Hence, even without random exploration of strategies [41], simple errors in the imitation process, finite β, are enough to overcome the unstable fixed point shown in Fig. 4.2 and reach a more cooperative basin of attraction on the righthand side of the gradient. As a result, for high values of risk and large, but finite, populations, cooperation is by far the strategy most favored by evolution irrespectively of the initial fraction of cooperators.

However, as mentioned in Chapter 2, fixation probability is not the whole story (see Sec. 2.3.3). For high intensities of selection or large populations, the fixation time can increase drastically. This is illustrated in Figs.4.3b,



(a) (b)

Figure 4.3: Effect of risk – fixation times
Evolutionary dynamics for different values of risk in finite populations. In panel (a), I show the fixation probabilities for different values of risk, r, as a function of the number of C's ($N = 50$, $c/b = 0.1$, $n = 6 = 2n_{pg}$, $\beta = 1.0$). In panels (b), I show the average number of generations (t_j/N) [52, 69] needed to fixate an initial fraction of 0.5 of cooperators, as a function of the intensity of selection β (top) and population size N (bottom). I consider the case of maximum risk ($r = 1$) for both (b) panels and $c/b = 0.1$, $n = 6 = 2n_{pg}$. Even if high risk can turn the fixation of cooperators almost certain (as shown in panel (a)), the time the population takes to reach such state can be arbitrarily long.

where is computed the average number of generations (t_i/N) needed to attain monomorphic cooperative state as a function of the intensity of selection and population size, starting from 50% of C's and D's for a dilemma with highest risk ($r = 1$). These panels clearly indicate that even if high risk can turn the fixation of cooperators almost certain (as shown in the left panel, Figs.4.3a), the time the population takes to reach such state can be arbitrarily long. While the computation of the fixation probabilities can be mathematically attractive, its relevance may be limited for large intensities of selection and/or large N. In other words, the stochastic information built in ϕ_i shows how unstable roots of g_i may be irrelevant; however, the lack of time information in ϕ_i ignores the key role played by the stable roots of g_i.

A proper alternative that overcomes the drawbacks identified in both ϕ_i and g_i consists in the analysis of the stationary distributions of the complete Markov chain p_i, of size $N + 1$ (see Chapter 2). Using the Fermi update rule in Eq.(3.1) I can compute the transition matrix. The transition matrix $T_{i'i}$ is tridiagonal and the stationary distribution is then obtained from the eigenvector corresponding to the eigenvalue 1 of $T_{i'i}$, or using Eq.(2.44). In Fig. 4.4 I show the stationary distributions for different values of risk, for a population of size $N = 50$ where $n = 2n_{pg} = 6$. While the finite population gradient of selection g_i shown in the inset exhibits a behavior qualitatively similar to $g(x)$ in Fig. 4.1, the stationary distributions show that the population spends most of the time in configurations where C's prevail, irrespective of the initial condition. This is a direct consequence of stochastic effects, which allow the "tunneling" through the coordination barrier associated with i_L, analogue of x_L, rendering such coordination barrier i_L irrelevant and turning cooperation into the prevalent strategy. On the other hand, the existence of a stable fixed root of g_i is triggered in p_i with a maximum at this position, unlike what one observes with ϕ_i.

Yet, until now the effect of the population size on the game itself remains uncharted. Fig. 4.5 is a plot of the roots of g_i as a function of the cost-to-risk ratio for different values of population size N. For large N the general picture described for infinite populations remains qualitatively valid. As before, two interior roots of g_i characterize the evolutionary dynamics of the population. However, the position of the interior fixed points can be profoundly altered by the population size. The range of i/N in which C's are advantageous is also strongly reduced for small populations. Moreover, while $\bar{x}$ (see Sec. 4.2) remains almost unchanged as one moves from infinite to finite populations, the critical $\bar{\gamma}$ is drastically reduced for small populations that, in turn, reduces the interval of cost-to-risk ratios for which a defection dominance dilemma is replaced by a combination of coordination and coexistence dilemmas. In other words, the smaller the population size the higher the perception of risk needed to achieve cooperation. The population size also plays an important role on the shape of the stationary distribution: In Fig. 4.5c I plot the stationary distribution for $r = 1$ and $c/b = 0.1$, for

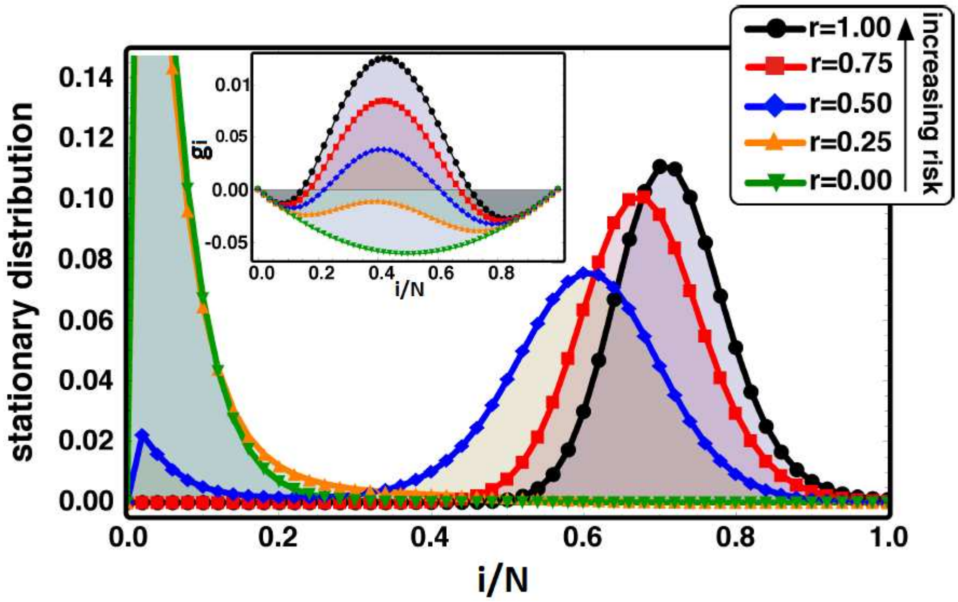
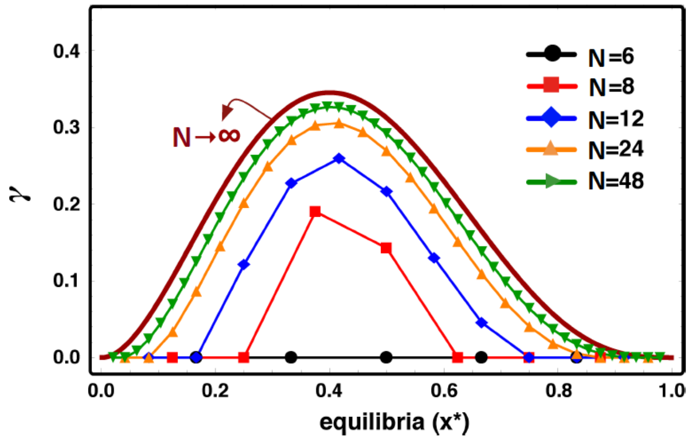


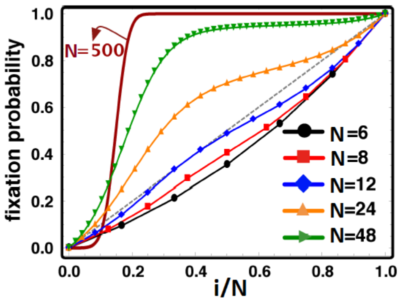
Figure 4.4: Effect of risk – stationary distributions
Prevalence of cooperation in finite populations. The main panel pictures the stationary distribution corresponding to the prevalence of each fraction of C's that emerges from the discrete gradient of selection g_i shown in inset. Whenever risk is high, stochastic effects turn collective cooperation into a pervasive behavior, rendering cooperation viable and favoring the overcome of coordination barriers, irrespective of the initial configuration ($N = 50$, $n = 6$, $n_{pg} = 3$, $c/b = 0.1$, $\mu = 0.005$).

different population sizes. Whenever the population size increases, a higher number of errors is needed to escape the equilibrium between C's and D's, leading the system to spend a higher fraction of time on the internal stable root of g_i.

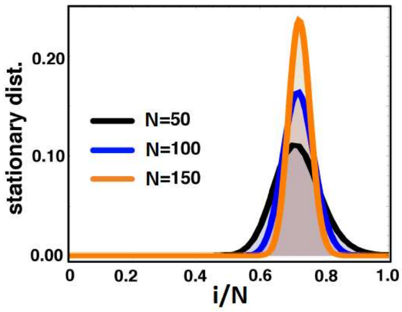
Naturally, the assessment of the effects of the population size should be carried out in combination with the number of parties involved in collective-risk dilemmas, i.e. the group size. Whether game participants are world citizens, world regions or country leaders, it remains unclear at which scale global warming should be tackled [57, 70]. Indeed, besides perception of risk, group size may play a pivotal role when maximizing the likelihood of reaching overall cooperation. As shown by the stationary distributions in Fig. 4.6, cooperation is better dealt with within small groups, with the proviso that for higher n_{pg}/n values, coordination is harder to attain, as shown by the position of the roots of g_i (see inset of Fig. 4.6).



(a)



(b)



(c)

Figure 4.5: Effect of risk – population size dependence

Population size dependence for $n = 6 = 2n_{pg}$. (a) Internal roots of the gradient of selection for different values of the cost-to-risk ratio and population sizes. (b) Fixation probabilities for different values of the population size for a fixed cost-to-risk ratio ($\gamma = 0.1$) as a function of the number of C's ($\beta = 5.0$). (c) I introduce a small mutation ($\mu = 0.005$) to show the stationary distribution for the same game parameters in (b) and different population sizes. As the population size increases, the system spends increasingly less time close to the monomorphic configurations. The three curves correspond, from top to bottom to, $N = 150, 100, 50$.

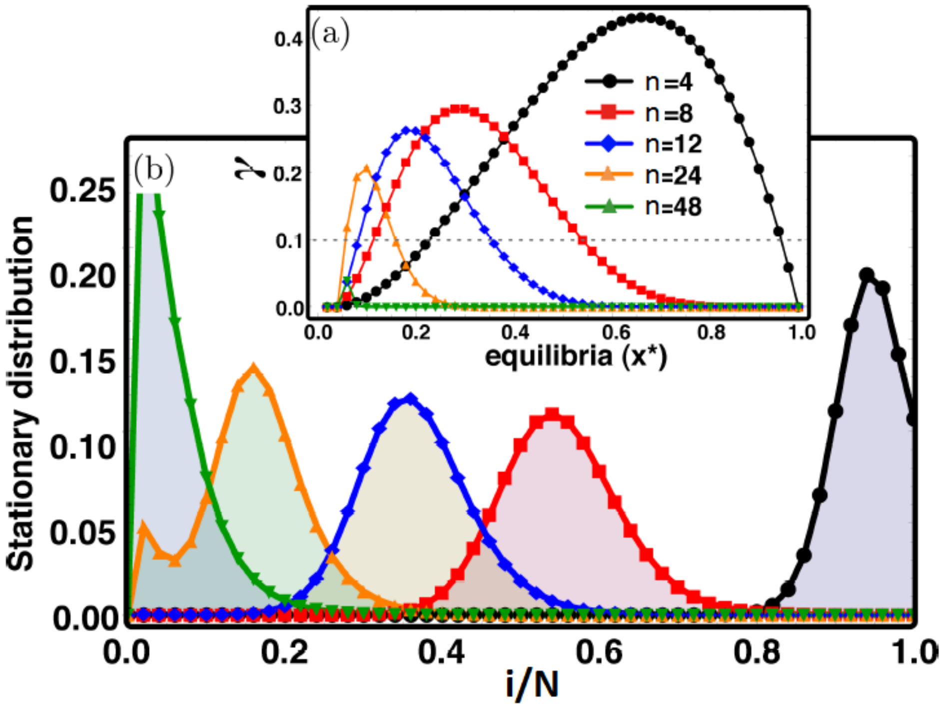


Figure 4.6: Effect of risk – group size dependence
Group size dependence for $n_{pg} = 3$. (a) Roots of the gradient of selection for different values of the cost-to-risk ratio and group sizes. (b) Stationary distribution for different group sizes and $c/\ br = 0.15$. Cooperation will be maximized in small groups, where the risk is high and goal achievement involves stringent requirements.

Chapter 5

Collective-risk Dilemmas and the effect of punishment

As often happens with human behavior, the problem is to find more efficient ways to promote cooperation. As discussed, artificial mechanisms must be used to promote the cooperation of all world leaders. Mechanisms in what group size of the decision-making process and perception of risk are concerned where described in the previous chapter. Another main tool often appointed is the idea of punishing those who do not contribute to the welfare of the planet. However, the way this punishment is done is still uncertain and it is hard to identify the benefits of some methods against others.

In this chapter I will study the effect of punishment comparing, against Defectors, punitive strategies with the cooperative strategy. Therefore, all the analysis can still be done using the one-dimensional analysis

Let me consider a CRD with two different types of punishment strategies, P, and, along with it, the other two, already studied, different behaviors: cooperation, C, and defection, D. Again, each of these types are defined by their action when interacting with $n-1$ other elements and, consequently, by their payoffs in such interaction.

$$\begin{aligned} P_{C\mathbf{j}} =& -c + b\,\Theta(j_C + j_P - n_{pg}) + \\ & + (1-r)b\,\Theta(n_{pg} - 1 - j_C - j_P) \end{aligned} \tag{5.1a}$$

$$P_{P\mathbf{j}} = P_{C\mathbf{j}} - p_t \tag{5.1b}$$

$$P_{D\mathbf{j}} = P_{C\mathbf{j}} + c - \Pi(\mathbf{i}, \mathbf{j}, n_p) \tag{5.1c}$$

In Eqs.(5.1), $\Theta(k) = \{^{0\ (k<0)}_{1\ (k\geq 0)}$ is the Heaviside function, n_{pg} a positive integer value not greater than n and r is real parameter between 0 and 1; the parameters c,p_t,p and b are all positive. If I am considering only C's and D's in the population $j_P = 0$, otherwise, if I consider only P's and D's,

$j_C = 0$. The interpretation is similar to what I have done before and is as follows.

C's and Punishers, P's, contribute a factor c, the *cost*, in order to reach a common partial goal - it represents the effort of cooperation - and, if there is enough contribution, $n_{pg}c$, everybody receives a *benefit* b, otherwise, they only receive that benefit with a probability $(1 - r)$. One calls r the *risk* of non-achievement and n_{pg} is the *public good game threshold*. Besides, P's also contribute with a *punishment tax*, p_t, to an institution which, whenever there is enough funding, $n_p p_t$, is able to efficiently punish D's by an amount Π, which corresponds to a *punishment function*. Notice that this *punishment* only occurs over a certain *punishment threshold*, n_p. However, the institution need not to be a global one, supported by all P's, that overviews all interactions but it can instead be local, created by the P's to overview the interaction with a given group of individuals. This will result in different *punishment functions*. If I want to use a local institution I pick Eq.(5.2), which means that a *punishment fine* p is applied to each D in the group whenever $i_P \geq n_p$, otherwise I choose Eq.(5.3), where punishment acts at the population level, applying a punishment fine p now to every D in the population, whenever $i_P \geq n_p$.

$$\Pi(\mathbf{i}, \mathbf{j}, n_p) = p\,\Theta(j_P - n_p) \tag{5.2}$$

$$\Pi(\mathbf{i}, \mathbf{j}, n_p) = p\,\Theta(i_P - n_p) \tag{5.3}$$

In the absence of P's, one recovers the n-person game of Chapter 4. As discussed, this model shows that the emerging dynamics depends on the perception of risk, population size and interaction group.

Now, I will consider the population size of the order of the number of countries involved in climate, several dozens, and the interacting group size one tenth. The different panels in Fig. 5.1 show the resulting stationary probability distribution function (PDF) from Eq.(2.4) together with the gradient of selection for two strategies in Eq.(3.4b) – C's and D's (left) and P's and D's (right). Once again, the maximum of the stationary distribution is nearly coincident with the configuration i_C^*, which makes an almost null gradient with negative slope. Consequently, population will spend most of the time around i_C^*, as it happened in the previous Chapter.

Fig.5.1 shows how risk (decreasing from top to bottom) still plays a crucial role in the overall population dynamics. The left panels reproduce the scenarios obtained in the absence of P's, obtained in Chapter 4, which will be used here as references, and reveal how sensitive the cooperation state is to risk. For a 50/50 chance of achieving the goal, despite having the maximum of the PDF in a cooperative state, one already sees a very strong negative gradient, directing the population towards the defectors side and making the right side tale of the distribution to fall rapidly. Has the risk decreases, one promptly watches the end of a cooperative population and, with a risk of 25%, all configurations tend to have less cooperators. When there is no risk, population receives a benefit whether they cooperate or not. Intuitively, it is of no use to cooperate and, consequently, population is driven to a configuration where everyone is a defector and will be there all the time.

Conversely, the right panels show the impact of punishment on the levels of cooperation – implemented here in the local institution version, see Eq.(5.2) – as P's replace C's in the entire population. In the absence of risk (bottom), the gradient of selection is nearly half as negative compared to the reference scenario. This means that the strength with which the population is driven into defection is smaller and, as a result, the stationary distribution grows a heavy tail towards punishment, meaning that the population actually spends a significant amount of time in configurations with less D's. This is a rather impressive result, revealing the power of punishment [71] in hindering (in this case) defection. As we increase risk (center and top panels) populations adopt more and more the punishment behavior.

An overall view of the results is provided in Fig.5.2 where I plot the internal roots of the gradient of selection for different values of r. I compare again the two strategies against D's: C's and P's with local institutions.

The CRD played between C's and D's, shows, as seen, two kinds of behaviors. In the first scenario, for low values of the perception of risk, the system is driven into a configuration in which defection dominates, apart from mutations, which generate a small amount of C's. As one increases the perception of risk, one reaches the second scenario that comprises two basins of attraction, where a local maximum appears now closer to full cooperation. Above a critical value of the perception of risk, the analogues of stable and unstable fixed points emerge, allowing the system to spend longer periods of time in more cooperative configurations. When I replace C's by P's, I also change the relative size of the two basins of attraction, in particular for low values of risk, as the critical perception of risk needed to create a cooperative basin of attraction occurs for lower values of r. Furthermore, with P's, the stable equilibrium where few D's co-exist with P's occurs for lower fraction of D's. Overall, this means that the population will spend a greater amount of the time in a more cooperative situation. Additionally, and compared to C's, P's also push the unstable fix point to lower fractions of D's, rendering collective coordination an easier task.

Finally, in Fig. 5.3 we adopt the same notation scheme of Fig. 5.2 to show the position of the internal roots of the gradient in terms of the *punishment fine*, p, and *punishment tax*,p_t. This analysis is repeated for different values of risk (low risk, $r = 0.0$, dots; intermediate risk, $r = 0.5$, triangles; high risk, $r = 1.0$, squares). In the top panels we see how the punishment tax affects the positions of the fixed points, for both low (left panel) and high (right panel) punishment fine. As expected, if the taxes for the maintenance of

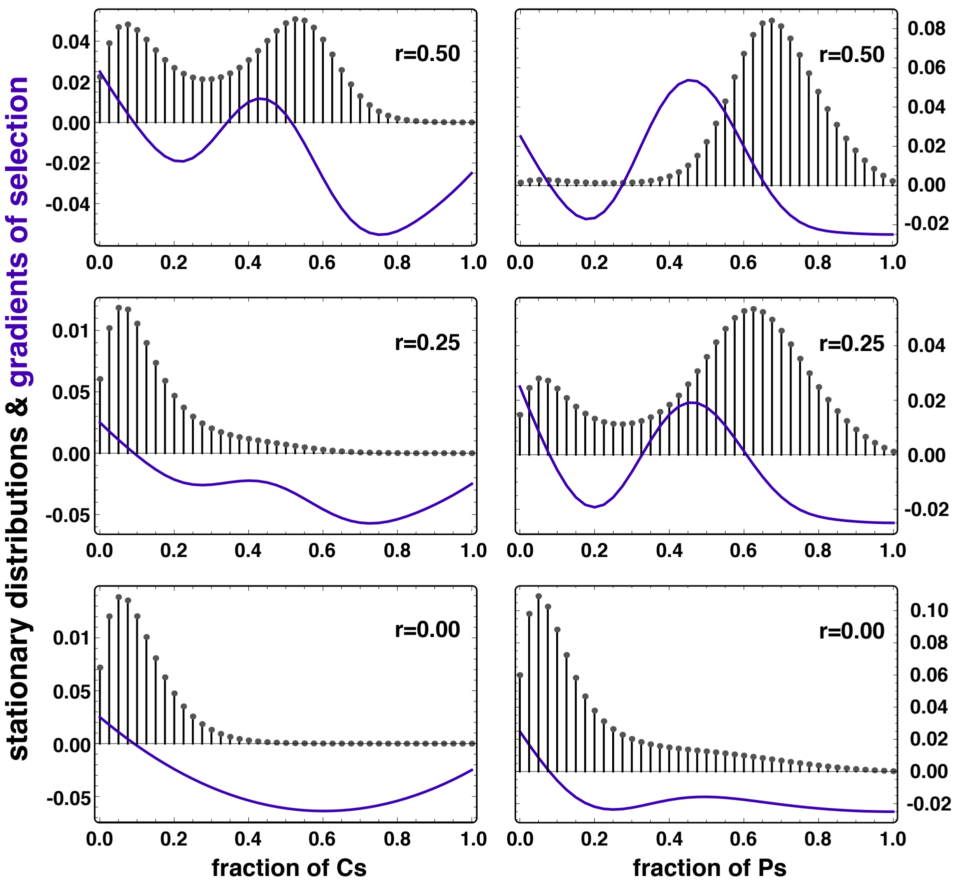


Figure 5.1: Effect of punishment – stationary distributions
The role of risk in populations made of C's and D's (left), and P's and D's (right). From the top to the bottom, each panel shows the stationary distributions and respective gradients of selection for $r = 0.5$, $r = 0.25$ and $r = 0$. Blue lines represent the gradient of selection and gray dots correspond to the stationary distribution function. ($N = 40$, $n = 10$, $n_{pg} = 5$, $b = 1$, $c = 0.1$, $p_t = 0.02$, $p = 0.1$, $n_p = 3$, $\mu = 0.025$ and $\beta = 5$).

institutions are low, a considerable amount of P's pervades; however, as we increase the tax, punishment eventually fades. When the punishment fine applied to the defectors is smaller, punishment vanishes for smaller values of the tax (left panel). In the bottom panels we show how the punishment fine affects the positions of the internal roots of , for both low (left panel) and high (right panel) punishment tax. If the tax for the punishment institution is low enough, a small punishment fine leads to the appearance of a coexistence root further from the full defection configuration. However, if the punishment fine is high, once again we regain the two different scenarios: for very low (or none) punishment the population falls into the tragedy

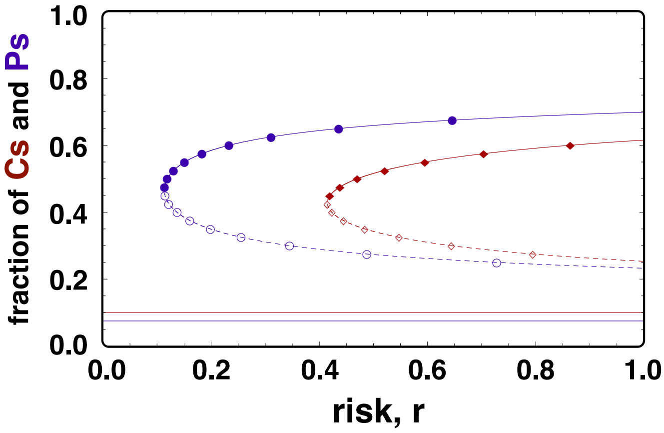


Figure 5.2: Effect of punishment – risk dependence
Interior roots of the gradient of selection for populations made of C's and D's (red lines and symbols), and P's and D's (blue lines and symbols). For each value of r, the full (empty) symbols represent the finite population analogues of stable (unstable) fixed points, that is, probability attractors (repellers). ($N = 40$, $n = 10$, $n_{pg} = 5$, $b = 1$, $c = 0.1$, $p_t = 0.02$, $p = 0.1$, $n_p = 3$, $\mu = 0.025$ and $\beta = 5$).

of the commons, whereas above a critical value of the punishment fine the coexistence point will arise. Both left and right panels show that a small increase on the punishment fine can drastically wipe-out defection. As a final point, all panels contain the location of the internal roots for the three values of risk indicated before, showing how important is the overall role of risk in the emergence collective action.

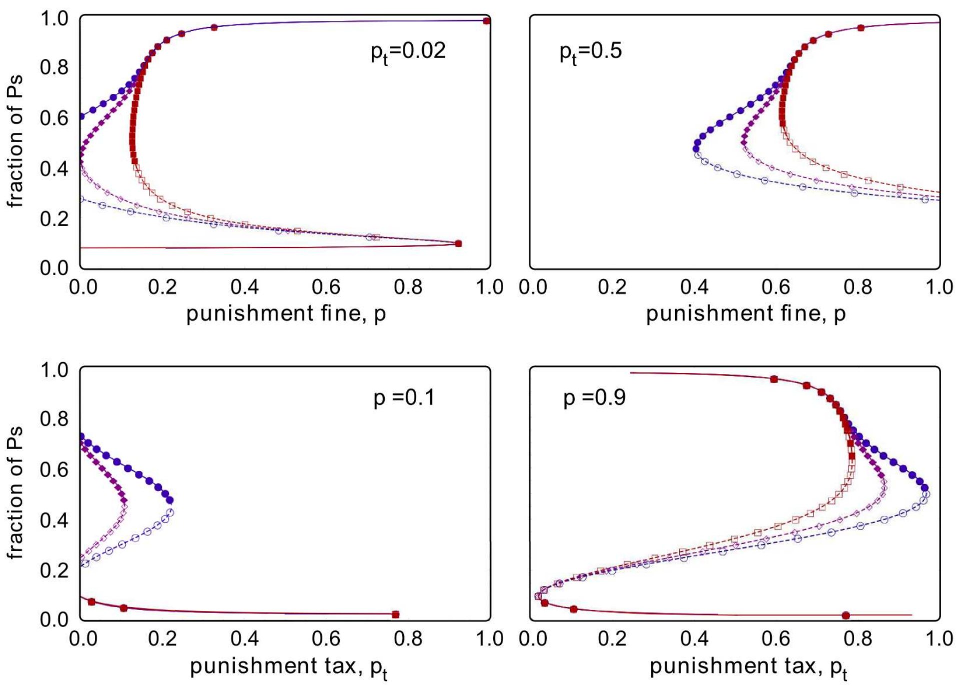


Figure 5.3: Effect of punishment – fine and tax dependence
Effect of punishment and sensitivity to risk. The top panels show the internal roots of the gradient as functions of p; the top left panel with low p_t and the bottom right panel with high p_t, respectively, $p_t = 0.02$ and $p_t = 0.50$. The bottom panels show the fixed point analogues as functions of the *punishment tax*, p_t; the top left panel with a low *punishment fine*, p, and the top right panel with a high p, respectively, $p = 0.1$ and $p = 0.9$. Dots, triangles and squares represent increasing values of risk, $r = 0$, $r = 0.5$, $r = 1$, respectively.($N = 40$,$n = 10$,$c = 0.1$,$b = 1.0$,$n_{pg} = 5$,$n_p = 3$,$\beta = 5$ and $\mu = 0.025$).

Chapter 6

Closer to real world summits

Now that I have shown the positive effect of punishment in a population, I can head closer into the real world summit meetings. To finish this thesis, I will explore the effects of global punishment institutions versus locally arranged ones and argue at which scale should punishment happen. As mentioned in the previous chapter, the institution need not be a global one (such as the United Nations), supported by all P's, that overviews all group-interactions in the population; it may also be a local, group-wide institution, created by the P's to enforce cooperation within a particular group of individuals. While the establishment of global institutions will depend on the total number of P's in the population, setting up local institutions relies solely on the number of P's within a group. Moreover, one does not expect that all the parties (e.g. countries, regions or cities [13]) will be willing to pay in order to punish others, despite being willing to undertake the necessary measures to mitigate the climate change effects (C's). In other words, one may expect to witness, in general, the three behaviors simultaneously in the population and, thus, one can only presume the system to evolve in such a way that cooperation naturally arises. Therefore, I will consider that the elements in the population are able to conceive the three kinds of behaviors I have used so far: C, D and P.

Notice that now, since I have $s + 1 = 3$ strategies, configurations will be settled on a bi-dimensional phase space. However, last end, we are *only* willing the planet salvation so it does not really matter if we save it with punishment or with cooperation alone. In this sense, I should be able to obtain results in the form of Fig.5.1 results, considering the overall fraction of non-defective strategies instead of the cooperative or punishing strategies by themselves.

Following what I have done so far, I consider a population of N individuals, who setup groups of size n, in which they engage in the CRD public goods game, being capable of adopting one of the three strategies: C, P and

D. The payoff of an individual playing in a group in which there are j_C C's, j_P P's and $j_D = n - j_C - j_D$ D's, are respectively written for C's, P's and D's in Eqs.(5.1), where now j_C and j_P can be simultaneously different from zero. If the overall contribution in the group is insufficient – that is, if the joint number of C's and P's in the group is below n_{pg} – everyone in that group will lose their remaining endowments with a probability r; otherwise, everyone will keep whatever they have. In addition, the fitness f_X of an individual adopting a given strategy, X, will be associated with the average payoff of that strategy in the population. This can be computed for a given strategy in a configuration $\mathbf{i} = i_C, i_P, i_D$ using a multivariate hypergeometric sampling, i.e. without replacement (see Sec. 3.2). The number of individuals adopting a given strategy will evolve in time according to a stochastic birth-death process combined with the pairwise comparison rule [40, 59], which describes the social dynamics of C's, P's and D's in a finite population (for details see Chapter 3).

I begin by analyzing global sanctioning institutions, that is, whenever there are enough individuals, population-wide, to setup such an institution that, in turn, will sanction all D's in the entire population of players. This is, in fact, the most natural scenario at present, given that countries typically make up the population of climate agreements, and hence the United Nations constitutes a natural candidate to supervise (and impose sanctions on) possible D's. Alternately, I also discuss the case of local sanctioning institutions. This latter case leads to a distributed scenario, as sanctions are the responsibility of a variety of institutions.

Figures 6.1 and 6.2 show representative examples of the behavioral dynamics of C's, D's and P's under global institutions (Fig. 6.1) and local institutions (Fig. 6.2), both for low (left panels) and high (right panels) values of the perception of risk. In the upper panels, we resort to a discrete three-strategy simplex (an equilateral triangle) at every point of which we have the relative frequencies of C's, D's and P's (summing up to one in every point), whereas each vertex is associated with a monomorphic configuration of the population. Given the stochastic nature of the dynamics, which includes mutations, the populations will never fixate in any of the three possible monomorphic configurations. This fact renders the stationary distribution the most appropriate tool to analyze the behavior of the population, since it provides information on the pervasiveness in time of each configuration. In the upper panels, the stationary distributions are shown for each configuration (each cell of the discrete simplex), using a grey-scale, where darker points indicate those configurations visited more often. Additionally, arrows in the simplex represent the most probable direction of evolution, obtained from the computation of the 2-dimensional gradient of selection (see Chapter 2). Arrows are drawn using a continuous color code (identified in the pictures) associated with the intensity of the gradient, i.e., the likelihood of such transitions.

In the lower panels, I show (with solid circles) the prevalence of non-defection in the population (that is, the combined fraction of C's and P's – those who contribute to the common good), obtained from projecting the information in the upper panels onto a one-dimensional simplex. In such a reduced configuration space (defection versus non-defection) the gradients of selection amount to a single line drawn in solid blue in the lower panels.

Let me start by considering the case of global institutions (Fig. 6.1). Whenever the population starts below n_p (the punishment threshold value) and in the absence of behavioral mutations, punishment will not occur, recovering a CRD played solely by C's and D's (see Chapter 4). When mutations allow the appearance of P's in the population, and depending on the actual perception of risk, the population may either be dominated by D's at low-risk (Fig. 6.1a), or D's will engage in a coordination-type game with non-D's (that is, C's and P's) as the perception of risk increases (Fig. 6.1b). When the composition of the population lies above the threshold line (indicated with orange arrows in the upper panel), D's get extinct in favor of P's (see Chapter 5), leading the population rapidly towards full cooperation, associated with the CP-edge of the simplex. In this situation, P's will be disadvantageous with respect to C's as they contribute to support an institution, which is now useless. Hence, the advantage of C's leads the population (slowly, as shown by the blue arrows) to a configuration that "falls below" the threshold line again. For low perception of risk, the only stable point of the dynamics below threshold will be full defection, as shown in Fig. 6.1a. Thus, as shown by the stationary distributions, the population will remain most of the time under widespread defection, with instances where C's and P's are able to co-exist due to mutations.

For high perception of risk, however, the situation is quite different, as shown in Fig. 6.1b. Indeed, in this case C's and D's easily manage to coexist for low fractions of P's, even for configurations below the threshold n_P, opening a window of opportunity for widespread cooperation to thrive. In other words, punishment will be effective with the crucial help of risk-perception, here mimicking the overall awareness of the danger associated with widespread defection.

Focusing now on local institutions, comparison between Fig. 6.2a (low risk-perception) and Fig. 6.2b (high risk-perception) shows that the role of the threshold line is not so pronounced in this case. Considering that we need the same fraction of P's to make the institution efficient, but now at the level of the group (and no longer at the level of the population), it is possible that some (though not all) games are played with a sufficient number of P's for sanctions to become effective. This leads the population as a whole to regimes in which it evolves towards widespread cooperation. Notably, in this case the population will then stabilize in configurations comprising a sizable amount of C's and enough P's to prevent D's from invading.

In this simple model, I was able to show that supervision is more efficient

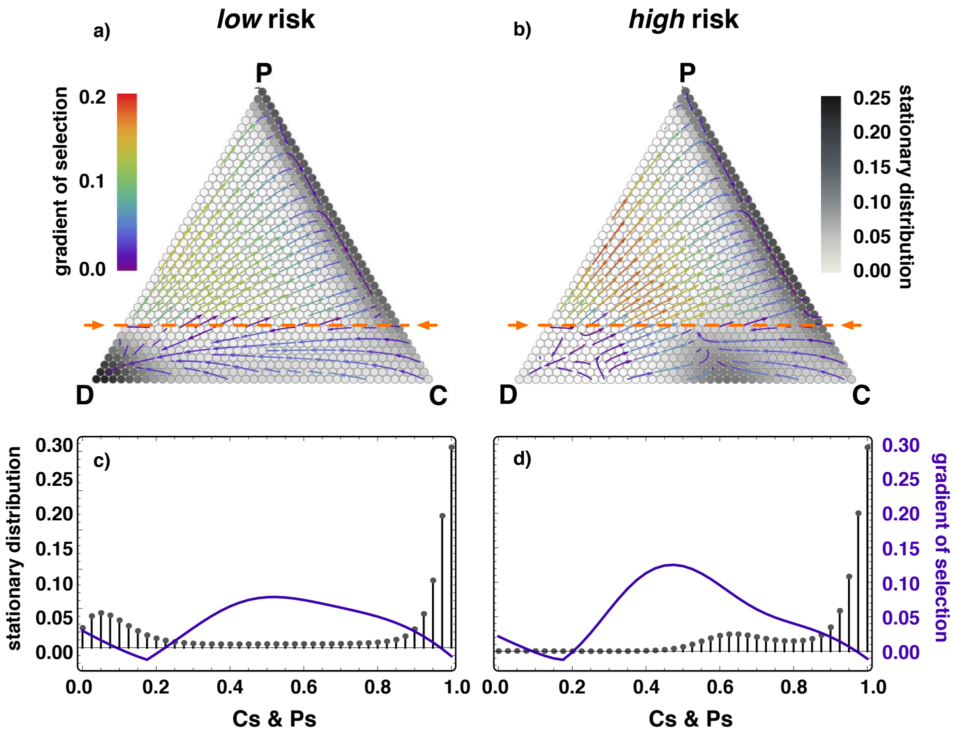


Figure 6.1: CRD with three strategies – global institutions
Collective Risk Dilemma (CRD) with three strategies: C's, P's and D's. Punishment is enacted by a Global, population-wide institution, who is responsible for fining all those who do not contribute to the pubic goods game in each group. The results depicted correspond to a population of size $N = 40$, where individuals interact in groups of size $n = 10$, with the benefit $b = 1$ and cost $c = 0.1$. The mutation probability is $\mu = 1/N$, whereas selection pressure is intermediate ($\beta = 5$); the threshold for punishment to become active (indicated by the horizontal line and arrows) is $n_P = 8$, that is, at least 20% of the population must be P's before D's are punished. Punishment tax is $p_t = 0.01$, whereas punishment for defecting is $p = 0.3$. In panel a) we show results for a low perception of risk ($r = 0.2$), whereas in panel b) risk perception is high ($r = 0.9$). In each full simplex shown in the upper panels, a gray scale is used to plot the stationary distribution, while the magnitude of the gradient of selection is represented in a continuous blue-yellow-red scale, where blue stands for the lowest intensities and red corresponds to the highest ones.

if it is accomplished in small groups. This avoids cycles of defection by increasing the number of interactions in which the defectors are punished. The fact that this happens for low r is of extreme importance, given that,

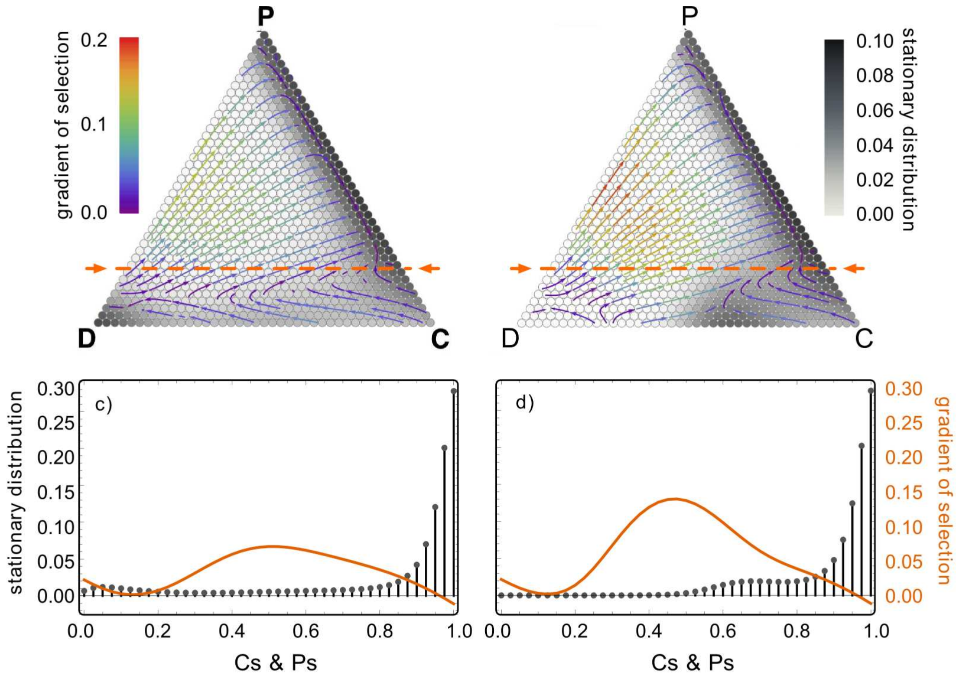


Figure 6.2: CRD with three strategies – local institutions
Collective Risk Dilemma with three strategies: C's, P's and D's. Same parameters and game settings as in Fig. 1, except that here, punishment is enacted by local institutions, who are responsible for fining all those (in the group) who do not contribute to the pubic goods game. I keep punishment threshold at $n_P = 2$, that is at least 20% of the individuals in each group must punish before institutional punishment becomes effective.

at present, the perception of risk regarding climate issues is low, mostly because the consequences of the actions required now will produce effects only in decades from now [2]. For high risk-perception, local institutions only enhance the positive prospects for cooperation already attained in the global institutions case.

Chapter 7

Conclusions

In dealing with the mitigation of climate change effects, we are facing a Public Goods dilemma in which the game must be played now despite the public good being associated with preventing future losses and, talking about environmental sustainability, one cannot overlook the uncertainty associated with collective investment. This, in turn, typifies many situations that humans did, do and will face throughout their history [72]. Here I propose a simple form to describe this problem and study its impact in behavioral evolution, a new n-person game where the risk of collective failure is explicitly introduced by means of a simple collective dilemma. Moreover, instead of resorting to complex and rational planning or rules, individuals revise their behavior by peer-influence, creating a complex dynamics akin to many evolutionary systems. With it, I obtained an unambiguous agreement with recent experiments [10], together with several concrete predictions. I do so in the framework of non-cooperative n-person evolutionary game theory, an unusual mathematical tool within the framework of modeling of political decision-making. This framework allowed to address the impact of risk in several configurations, from large to small groups, from deterministic towards stochastic behavioral dynamics and also the effect of punishment.

Overall, I have shown how the emerging behavioral dynamics depends heavily on the perception of risk. The impact of risk is enhanced in the presence of small behavioral mutations and errors and whenever global coordination is attempted in a majority of small groups under stringent requirements to meet co-active goals. This result calls for a reassessment of policies towards the promotion of public endeavors: instead of world summits, decentralized agreements between smaller groups (small n), possibly focused on region-specific issues, where risk is high and goal achievement involves tough requirements (large relative n_{pg}), are prone to significantly raise the probability of success in coordinating to tame the planet's climate.

In addition, I have shown how individuals may effectively self-organize their actions towards cooperation, by creating community enforcement institutions able to punish those who row against collective interest [31, 11]. Moreover, I offer insights on the scale at which such institutions should be implemented, suggesting that a decentralized, polycentric, bottom-up approach, involving multiple institutions instead of a single global one, provides better conditions both for cooperation to thrive and for ensuring the maintenance of such institutions. This result is particularly relevant whenever perception of risk of collective disaster, alone, is not enough to provide the means to achieve a cohesive configuration – in this case, local sanctioning institutions may provide an escape hatch to the otherwise tragedy of the commons that humanity is falling into.

This model provides a "bottom-up" approach to the problem, in which collective cooperation is easier to achieve in a distributed way, eventually involving regions, cities, NGOs and, ultimately, all citizens. Moreover, by promoting regional or sectorial agreements, we are opening the door to the diversity of economic and political structure of all parties, which, as showed before [34, 73] can be beneficial to cooperation.

Naturally, I am aware of the many limitations of a bare model such as this, in which the complexity of human interactions has been overlooked. From higher levels of information, to non-binary investments, additional layers of realism can be introduced in the model. Moreover, from a mathematical perspective, several extensions and complex aspects common to human socio-economical systems could be further explored [74, 75, 76, 77]. On the other hand, the simplicity of the dilemma introduced here, makes it generally applicable to other problems of collective cooperative action, which will emerge when the risks for the community are high, something that repeatedly happened throughout human history [78, 79], from ancient group hunting to voluntary adoption of public health measures [53, 80, 81]. Similarly, other cooperation mechanisms [11, 17, 18, 20, 24, 25, 26, 27, 28] known to encourage collective action, may further enlarge the window of opportunity for cooperation to thrive.

While most causes of climate change result from the combined action of all inhabitants of our planet, the solutions for such complex and global dilemma may be easier to achieve at a much smaller scale [14]. In light of these results, the widely-repeated motto "Think globally but act locally" would hardly appear more appropriate.

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